

A CLASSIFICATION OF THE IRREDUCIBLE MOD- p REPRESENTATIONS OF $U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$

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ABSTRACT. Let p be an odd prime number. We classify all smooth irreducible mod- p representations of the unramified unitary group $U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ in two variables. We then investigate Langlands parameters in characteristic p associated to $U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, and propose a correspondence between certain equivalence classes of Langlands parameters and certain isomorphism classes of semisimple L -packets on $U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$.

CONTENTS

1. Introduction	1
2. Notation	4
2.1. General Notation	4
2.2. Bruhat Decompositions	6
3. Hecke Algebras	7
3.1. Preliminaries	7
3.2. Pro- p -Iwahori-Hecke Algebra	7
3.3. Decomposition with Idempotents	8
4. Nonsupercuspidal Representations	9
5. Supercuspidal Representations	12
5.1. The Group $GL_2(\mathbb{Q}_p)$	13
5.2. The Group $SL_2(\mathbb{Q}_p)$	13
5.3. The Group $U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$	14
5.4. L -packets	17
6. Galois Groups and Representations	19
6.1. Galois Groups	19
6.2. L -groups	20
6.3. Langlands Parameters for $U(1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$	22
6.4. Langlands Parameters for G	23
6.5. Remarks	28
References	29

1. INTRODUCTION

Recently, the mod- p representation theory of p -adic reductive groups has garnered a great deal of attention as a result of its roles in the mod- p and p -adic Local Langlands Programs. The expectation is that there exists a matching between (packets of) smooth mod- p representations of a p -adic reductive group and certain Galois representations. Representations of

the group $\mathrm{GL}_2(\mathbb{Q}_p)$ have been widely studied and analyzed, and a mod- p Local Langlands Correspondence has been established by Breuil ([8]) based on the explicit determination of the irreducible mod- p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Moreover, this correspondence is compatible with the p -adic Local Langlands Correspondence (cf. [9]; see also [10], [12], [14], [19], [20], [24]).

The smooth irreducible mod- p representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ have recently been classified by Abdellatif in [2], by examining restrictions of the irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. This allowed her to take the first steps towards a mod- p Local Langlands Correspondence for $\mathrm{SL}_2(\mathbb{Q}_p)$. In addition, her results are the first to consider a mod- p Local Langlands Correspondence with L -packets.

Several aspects of the mod- p representation theory of unitary groups have already been considered by Abdellatif in [1], and by the author and Xu in [21]. In the present article, we utilize the work of Breuil and Abdellatif to investigate the smooth irreducible mod- p representations of the unitary group $\mathrm{U}(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, where \mathbb{Q}_{p^2} denotes an unramified quadratic extension of the field of p -adic numbers \mathbb{Q}_p . The irreducible subquotients of parabolically induced representations of G have been classified by Abdellatif in [1]. We shall be interested in representations which do not arise in this way, which we will refer to as *supercuspidal* representations (we will comment on terminology at the end of this introduction). These representations are the ones which are expected to play a central role in a potential Local Langlands Correspondence. We now describe our method which classifies these representations.

We begin in a more general context. Denote by F a nonarchimedean local field of residual characteristic p , and E an unramified quadratic extension. Let G be the group $\mathrm{U}(1, 1)(E/F)$, $G_S = \mathrm{SU}(1, 1)(E/F)$ its derived subgroup, and $I_S(1)$ the unique pro- p Sylow subgroup of the standard Iwahori subgroup of G_S . The *pro- p -Iwahori-Hecke algebra* $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ is the convolution algebra of compactly supported, $\overline{\mathbb{F}}_p$ -valued functions on the double coset space $I_S(1) \backslash G_S / I_S(1)$. As $G_S \cong \mathrm{SL}_2(F)$, the structure and properties of $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ are well-understood (cf. [1], [30]). In particular, a classification of finite-dimensional simple right modules for $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ is known. We review the necessary results in Chapter 3.

The results of [1] provide a classification of smooth irreducible nonsupercuspidal representations of any connected quasi-split reductive group of relative rank 1. We make the computations explicit in Section 4, and obtain a precise description of all irreducible nonsupercuspidal representations of G (Theorem 4.3). We then investigate the behavior of irreducible representations upon restriction to the derived subgroup G_S . From the classification of irreducible nonsupercuspidal representations of G and G_S , we obtain a result on the restrictions of supercuspidal representations of G (Proposition 4.7).

Next, we specialize to the case where $F = \mathbb{Q}_p$ and $E = \mathbb{Q}_{p^2}$. Under these assumptions, the smooth irreducible supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ have been classified by Breuil in [8]. They are all of the form

$$\pi(r, 0, \chi) = \chi \circ \det \otimes \frac{\mathrm{c-ind}_{\mathbb{Q}_p^\times \mathrm{GL}_2(\mathbb{Z}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} (\mathrm{Sym}^r(\overline{\mathbb{F}}_p^2))}{(T_r)},$$

where $0 \leq r \leq p-1$, $\chi : \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{F}}_p^\times$ is a smooth character, and T_r is a generator of a certain spherical Hecke algebra. Results of Abdellatif ([2]) show that one can then obtain the

irreducible supercuspidal representations of $SL_2(\mathbb{Q}_p)$ by restriction; we have

$$\pi(r, 0, \chi)|_{SL_2(\mathbb{Q}_p)} \cong \pi_r \oplus \pi_{p-1-r},$$

where π_r is an irreducible supercuspidal $SL_2(\mathbb{Q}_p)$ -subrepresentation of $\pi(r, 0, 1)$ generated by a distinguished pro- p -Iwahori-fixed vector. In light of the isomorphism $SL_2(\mathbb{Q}_p) \cong SU(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) = G_S$, we view the representations π_r as smooth representations of G_S .

Using a cohomological argument, we show in Section 5 that the representations π_r lift to smooth irreducible representations of G , which we denote

$$\omega^k \circ \det \otimes \pi_r$$

for $0 \leq r \leq p-1$ and $0 \leq k < p+1$ (see Definition 5.7). Moreover, we show that every smooth irreducible supercuspidal representation of $U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ is of this form (Theorem 5.9), and thereby obtain a classification of all smooth irreducible representations (Corollary 5.10). To conclude, we arrange the irreducible representations into sets called L -packets, and determine the L -packets on $U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ explicitly.

In the final section, we define the relevant Galois groups and L -groups attached to G . Our definitions are adapted from the complex setting (see [25] for the classical definitions). Thus, we are led to investigate Langlands parameters associated to the group G , that is, homomorphisms

$$\varphi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \longrightarrow {}^L G = \text{GL}_2(\overline{\mathbb{F}_p}) \rtimes \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p),$$

such that the composition of φ with the canonical projection ${}^L G \longrightarrow \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is the identity map of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Our first (somewhat surprising) result in this direction is Proposition 6.14, which asserts that there do not exist any parameters φ such that the Galois representation associated to $\varphi|_{\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2})}$ is irreducible.

The aforementioned result suggests that the Langlands parameters associated to G which are of interest are “reducible” in some sense. We therefore consider two intermediate L -groups ${}^L J$ and ${}^L T$, where J is the unique *elliptic endoscopic group* associated to G , and T is the maximal torus of G . The Langlands parameters we examine take the following form:

$$\varphi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \longrightarrow {}^L J = (\overline{\mathbb{F}_p}^\times \times \overline{\mathbb{F}_p}^\times) \rtimes \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \hookrightarrow {}^L G$$

$$\psi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \longrightarrow {}^L T = (\overline{\mathbb{F}_p}^\times \times \overline{\mathbb{F}_p}^\times) \rtimes \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \hookrightarrow {}^L G$$

We stress that the action of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on $\overline{\mathbb{F}_p}^\times \times \overline{\mathbb{F}_p}^\times$ for the group ${}^L J$ is *not the same* as the action for the group ${}^L T$. We classify all such parameters in Corollary 6.16 and Proposition 6.20, and determine the possible equivalences among the parameters (Lemmas 6.18, 6.22, and 6.23).

In the complex setting, the parameters coming from ${}^L J$ play a pivotal role in the representation theory of G . In the characteristic p setting, they remain of particular importance. We denote the parameters factoring through ${}^L J$ by $\varphi_{k,\ell}$, where $0 \leq k, \ell < p+1$ (see Definition 6.17 for their precise description). Using equivalences between the parameters $\varphi_{k,\ell}$ and the classification of L -packets, we obtain the following:

Theorem (Corollary 6.19). *Suppose $0 \leq k, \ell < p+1$ and $k \neq \ell$. There exists a bijection between equivalence classes of Langlands parameters coming from the group ${}^L J$ and L -packets of irreducible supercuspidal representations of the group $G = U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, given by*

$$\varphi_{k,\ell} \longleftrightarrow \{\omega^\ell \circ \det \otimes \pi_{[k-\ell-1]}, \omega^k \circ \det \otimes \pi_{[\ell-k-1]}\},$$

where $[k - \ell - 1]$ (resp. $[\ell - k - 1]$) denotes the unique integer between 0 and $p - 1$ equivalent to $k - \ell - 1$ (resp. $\ell - k - 1$) modulo $p + 1$. Moreover, this bijection is compatible with twisting on both sides.

Finally, we extend this bijection to include semisimple nonsupercuspidal representations (Definition 6.24), using the parameters coming from ${}^L T$. This allows us to make explicit a case of *endoscopic transfer* from irreducible representations of J to semisimple L -packets on G (see the remarks following Definition 6.24).

Remark on Terminology. We briefly address our choice of nomenclature. In the mod- p representation theory of p -adic groups, the representations dubbed supersingular are the ones which play a crucial role. The notion of supersingularity was introduced by Barthel and Livné ([3] and [4]) in their classification of smooth irreducible mod- p representations of $\mathrm{GL}_2(F)$. A smooth representation of $\mathrm{GL}_2(F)$ is called *supersingular* if a certain operator of a spherical Hecke algebra acts by zero, while it is called *supercuspidal* if it is not a subquotient of a parabolically induced representation. Theorems 33, 34, and Corollary 36(1) of [3] show that a smooth representation of $\mathrm{GL}_2(F)$ admitting a central character is supercuspidal if and only if it is supersingular. In the present case, the study of spherical Hecke algebras for nonsplit groups has been initiated by Henniart and Vignéras in [15] and by Herzig in [17]. Based on results contained in [16] and [18], it is very likely that the notions of supersingularity and supercuspidality coincide. As we have not attempted to make an in-depth study of spherical Hecke algebras for $\mathrm{U}(1, 1)(E/F)$, we feel more comfortable using the terminology of supercuspidal and nonsupercuspidal representations. We hope to address these questions in future work.

Acknowledgements. I would like to thank my advisor Professor Rachel Ollivier, for many enlightening discussions throughout the course of working on this article, as well as her encouragement and advice. I would also like to thank Professor Shaun Stevens for several helpful suggestions, and for reading a (very) rough first draft. During the preparation of this article, support was provided by NSF Grant DMS-0739400.

2. NOTATION

2.1. General Notation. Fix a prime number p greater than 2, and let F be a nonarchimedean local field of residual characteristic p . Denote by \mathfrak{o}_F its ring of integers, and by \mathfrak{p}_F the unique maximal ideal of \mathfrak{o}_F . Fix a uniformizer ϖ_F and let $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ denote the (finite) residue field. The field k_F is a finite extension of \mathbb{F}_p of size $q = p^f$. We fix also a separable closure \overline{F} of F , and let $k_{\overline{F}}$ denote its residue field.

Let E denote the unique unramified extension of degree 2 in \overline{F} . The ring of integers of E is denoted \mathfrak{o}_E , and \mathfrak{p}_E is its unique maximal ideal. Since E is unramified, we may and do take $\varpi_E = \varpi_F =: \varpi$ as our uniformizer. The residue field of E is $k_E = \mathfrak{o}_E/\mathfrak{p}_E$; it is a degree 2 extension of k_F . Let $\iota : k_{\overline{F}} \xrightarrow{\sim} \overline{\mathbb{F}}_p$ denote a fixed isomorphism, and assume that every $\overline{\mathbb{F}}_p^\times$ -valued character factors through ι . We identify k_F and k_E with \mathbb{F}_q and \mathbb{F}_{q^2} , respectively, using the isomorphism ι . We shall denote by \mathbb{Q}_p the field of p -adic numbers, and by \mathbb{Q}_{p^2} its unique unramified quadratic extension in a fixed algebraic closure $\overline{\mathbb{Q}}_p$.

We write $E = F(\sqrt{\epsilon})$, where $\epsilon \in \mathfrak{o}_F^\times$ is some fixed but arbitrary nonsquare unit. We let $x \mapsto \bar{x}$ denote the nontrivial Galois automorphism of E fixing F , and define $U(1)(E/F)$ to be the kernel of the norm map

$$\begin{aligned} N_{E/F} : E^\times &\longrightarrow F^\times \\ x &\longmapsto x\bar{x}. \end{aligned}$$

We denote by ϑ a fixed representative of the nontrivial coset in $\mathfrak{o}_E^\times/\mathfrak{o}_F^\times U(1)(E/F)$.

Denote by G the F -rational points of the algebraic group $U(1,1)$. We perform our computations using the following realization of G : let V denote a two-dimensional hyperbolic plane over E . We identify V with E^2 by a choice of basis, and for $\vec{x} = (x_1, x_2)^\top, \vec{y} = (y_1, y_2)^\top \in V$ the nondegenerate Hermitian form $\langle \cdot, \cdot \rangle$ is given by

$$\langle \vec{x}, \vec{y} \rangle = \bar{x}_1 y_2 + \bar{x}_2 y_1.$$

Letting

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

our form is represented by $\langle \vec{x}, \vec{y} \rangle = \vec{x}^* s \vec{y}$, where m^* denotes the conjugate transpose of a matrix m with coefficients in E . With this notation, we have

$$G = \{g \in \mathrm{GL}_2(E) : g^* s g = s\}.$$

Let K denote the maximal compact subgroup of G given by

$$K := \mathrm{GL}_2(\mathfrak{o}_E) \cap G,$$

and let

$$K_1 := \begin{pmatrix} 1 + \mathfrak{p}_E & \mathfrak{p}_E \\ \mathfrak{p}_E & 1 + \mathfrak{p}_E \end{pmatrix} \cap G$$

denote its maximal normal pro- p subgroup. The group K is a representative of one of two conjugacy classes of maximal compact subgroups of G ([28], Sections 2.10 and 3.2). We define

$$\Gamma := K/K_1 \cong U(1,1)(\mathbb{F}_{q^2}/\mathbb{F}_q).$$

The Iwahori subgroup I is defined as the preimage under the quotient map $K \rightarrow \Gamma$ of the Borel subgroup of upper triangular matrices in Γ . We denote by $I(1)$ the unique pro- p -Sylow subgroup of I , which is the preimage of the upper triangular unipotent elements of Γ . Explicitly, we have

$$I := \begin{pmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{o}_E^\times \end{pmatrix} \cap G, \quad I(1) := \begin{pmatrix} 1 + \mathfrak{p}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & 1 + \mathfrak{p}_E \end{pmatrix} \cap G.$$

Let B denote the Borel subgroup of upper triangular elements of G , U its unipotent radical, and U^- the opposite unipotent. The subgroups U and U^- are both isomorphic to the additive group F . We define

$$u(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad u^-(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

where $x \in E$ satisfies $x + \bar{x} = 0$. We have $u(x)^{-1} = u(\bar{x})$, $u^-(x)^{-1} = u^-(\bar{x})$.

We let G_S denote the derived subgroup of G . For any subgroup J of G , we let J_S denote its intersection with G_S . We have $G_S = \mathrm{SU}(1,1)(E/F) \cong \mathrm{SL}_2(F)$, the latter isomorphism given by conjugation by the element

$$\begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(E).$$

We shall exploit this isomorphism to give a classification of the smooth irreducible representations of $U(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$.

We define the following distinguished elements of $GL_2(E)$:

$$\begin{aligned} s &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & s' &:= \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{pmatrix}, \\ n_s &:= \begin{pmatrix} 0 & -\sqrt{\epsilon}^{-1} \\ \sqrt{\epsilon} & 0 \end{pmatrix}, & n_{s'} &:= \begin{pmatrix} 0 & -\varpi^{-1}\sqrt{\epsilon}^{-1} \\ \varpi\sqrt{\epsilon} & 0 \end{pmatrix}, \\ \alpha &:= s's = \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & \varpi \end{pmatrix}, & \alpha^{-1} &:= ss' = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix}, \\ \beta &:= \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}, & \theta &:= \begin{pmatrix} \vartheta & 0 \\ 0 & \vartheta^{-1} \end{pmatrix}. \end{aligned}$$

2.2. Bruhat Decompositions. The maximal torus T of G consists of all elements of the form

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix},$$

with $a \in E^\times$. Note that T is not split over F . The maximal torus of G_S is $T_S = T \cap G_S$; it consists of all elements of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

with $a \in F^\times$. The center Z of G is given by the subgroup of elements

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

with $a \in U(1)(E/F)$.

Let

$$T_0 := T \cap K, \quad T_1 := T \cap K_1, \quad H := T_0/T_1 \cong I/I(1) \cong \mathbb{F}_{q^2}^\times.$$

We will identify the characters of H and those of $I/I(1)$. We will also identify \mathbb{F}_{q^2} with the image of the Teichmüller lifting map $[\cdot] : \mathbb{F}_{q^2} \rightarrow \mathfrak{o}_E$ when convenient.

Let N denote the normalizer of T in G . Then the affine Weyl group W_{aff} is defined as N/T_0 , and the finite Weyl group W is defined as N/T . The group W_{aff} is a Coxeter group, generated by the classes of the two reflections s and s' . We have a decomposition $G = INI$, where two cosets InI and $In'I$ are equal if and only if n and n' have the same image in W_{aff} . This yields the Bruhat decomposition for the BN pair (I, N) :

$$G = \bigsqcup_{w \in W_{\text{aff}}} IwI;$$

here we engage in the standard abuse of notation, letting IwI denote $I\dot{w}I$ for any preimage \dot{w} of w in N . We will take as our double coset representatives the elements $\alpha^n, n_s \alpha^n$, for $n \in \mathbb{Z}$.

Similarly, we have a Bruhat decomposition for the group G_S :

$$G_S = \bigsqcup_{w \in N_S/T_{0,S}} I_S w I_S.$$

In light of the isomorphism $N/T_0 \cong N_S/T_{0,S}$, we will take the elements $\alpha^n, n_s \alpha^n$, for $n \in \mathbb{Z}$ as our double coset representatives in G_S .

3. HECKE ALGEBRAS

In the course of determining the smooth irreducible representations of $U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, we shall make essential use of the pro- p -Iwahori-Hecke algebra of the derived subgroup $SU(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. We collect the relevant results here.

3.1. Preliminaries. For this subsection only, let G denote any locally profinite group. We shall be interested in the category $\mathfrak{Rep}_C(G)$ of smooth representations of G over C , where C denotes an algebraically closed field. We briefly recall some preliminary terminology. Let J be a closed subgroup of G , and let (σ, V_σ) be a smooth C -representation of J (meaning that stabilizers are open). We denote by $\text{ind}_J^G(\sigma)$ the space of functions $f : G \rightarrow V_\sigma$ such that $f(jg) = \sigma(j)f(g)$ for $j \in J, g \in G$, and such that the action of G given by right translation is smooth (meaning that there exists some open subgroup J' , depending on f , such that $f(gj') = f(g)$ for every $j' \in J', g \in G$). We let $\text{c-ind}_J^G(\sigma)$ denote the subspace of $\text{ind}_J^G(\sigma)$ spanned by functions whose support in $J \backslash G$ is compact. These functors are called *induction* and *compact induction*, respectively. We will mostly be concerned with the cases when J is a compact open subgroup, or when J is a parabolic subgroup of a reductive algebraic group G .

3.2. Pro- p -Iwahori-Hecke Algebra. Let π be a smooth C -representation of the group $G_S = SU(1,1)(E/F)$. Frobenius Reciprocity for compact induction gives

$$\pi^{I_S(1)} \cong \text{Hom}_{I_S(1)}(1, \pi|_{I_S(1)}) \cong \text{Hom}_{G_S}(\text{c-ind}_{I_S(1)}^{G_S}(1), \pi),$$

where 1 denotes the trivial character of $I_S(1)$. The *pro- p -Iwahori-Hecke algebra*

$$\mathcal{H}_C(G_S, I_S(1)) := \text{End}_{G_S}(\text{c-ind}_{I_S(1)}^{G_S}(1))$$

is the algebra of G_S -equivariant endomorphisms of the universal module $\text{c-ind}_{I_S(1)}^{G_S}(1)$. This algebra has a natural right action on $\text{Hom}_{G_S}(\text{c-ind}_{I_S(1)}^{G_S}(1), \pi)$ by pre-composition, which induces a right action on $\pi^{I_S(1)}$. In this way, we obtain the functor of $I_S(1)$ -invariants, $\pi \mapsto \pi^{I_S(1)}$, from the category of smooth C -representations of G_S to the category of right $\mathcal{H}_C(G_S, I_S(1))$ -modules.

By adjunction, we have a natural identification

$$\mathcal{H}_C(G_S, I_S(1)) \cong \text{c-ind}_{I_S(1)}^{G_S}(1)^{I_S(1)},$$

so we may view endomorphisms of $\text{c-ind}_{I_S(1)}^{G_S}(1)$ as compactly supported functions on G_S which are $I_S(1)$ -biinvariant. This leads to the following definition.

Definition 3.1. Let $g \in G_S$. We let $T_g \in \mathcal{H}_C(G_S, I_S(1))$ denote the endomorphism of $\text{c-ind}_{I_S(1)}^{G_S}(1)$ corresponding by adjunction to the characteristic function of $I_S(1)gI_S(1)$; in particular, T_g maps the characteristic function of $I_S(1)$ to the characteristic function of $I_S(1)gI_S(1)$.

From this definition it is clear that $T_g = T_{g'}$ if and only if $I_S(1)gI_S(1) = I_S(1)g'I_S(1)$; moreover, since $W_{\text{aff}} = N_S/T_{0,S}$ is a set of representatives for the double coset space $I_S \backslash G_S / I_S$, the group $N_S/T_{1,S}$ gives a set of representatives for $I_S(1) \backslash G_S / I_S(1)$. We therefore only consider the operators T_n , where n is a representative of a coset in $N_S/T_{1,S}$. These operators give a basis for $\mathcal{H}_C(G_S, I_S(1))$ as a vector space over C . Using the isomorphisms above, we see that if π is a smooth C -representation of G_S , $v \in \pi^{I_S(1)}$, and $n \in N_S$, then

$$(1) \quad v \cdot T_n = \sum_{u \in I_S(1) \backslash I_S(1)nI_S(1)} u^{-1} \cdot v = \sum_{u \in I_S(1)/I_S(1) \cap n^{-1}I_S(1)n} un^{-1} \cdot v.$$

3.3. Decomposition with Idempotents. We assume henceforth that $C = \overline{\mathbb{F}}_p$. Let $H_S := T_{0,S}/T_{1,S} \cong \mathbb{F}_q^\times$, and denote by $\widehat{H_S}$ the group of all $\overline{\mathbb{F}}_p^\times$ -valued characters of H_S . For $r \in \mathbb{Z}$, we denote by χ_r the character of H_S given by

$$\chi_r \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^r,$$

where $a \in \mathbb{F}_q^\times$. Every element of $\widehat{H_S}$ is of the form χ_r with $0 \leq r < q-1$.

Definition 3.2. For an $\overline{\mathbb{F}}_p$ -character χ_r of H_S , we define

$$e_r = |H_S|^{-1} \sum_{h \in H_S} \chi_r(h) T_h,$$

where T_h denotes the operator T_{t_0} for any preimage of t_0 of h in $T_{0,S}$.

The following properties of the operators e_r follow readily from the orthogonality relations of characters:

- $e_r e_r = e_r$;
- $e_r e_{r'} = 0$ for $\chi_r \neq \chi_{r'}$;
- $\text{id}_{\text{c-ind}_{I_S(1)}^{G_S}(1)} = \sum_{r=0}^{q-2} e_r$.

Using these orthogonal idempotents e_r , we obtain the following structure theorem. We note that the results of [30] apply in this setting, since the group G_S is *split*.

Theorem 3.3. *The operators T_{n_s} , $T_{n_{s'}}$ and e_r , for $0 \leq r < q-1$, generate $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ as an algebra.*

Proof. The claim follows from (the remark following) Theorem 1 of [30], along with Fourier inversion on the elements e_r . \square

In the study of Hecke modules over fields of characteristic p , the notion of supersingularity plays a prominent role. We recall the definition here.

Definition 3.4 ([30], Definition 3). Let M be a nonzero simple right $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ -module which admits a central character. We say M is *supersingular* if every element of the center of $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ which is of “positive length” acts by 0. For the precise notion of “positive length,” see the discussion preceding Definition 2 (*loc. cit.*).

The finite-dimensional simple right $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ -modules have been classified in Chapitre 6 of [1]. The supersingular modules take on a particularly simple form:

Proposition 3.5. *The supersingular $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ -modules are one-dimensional. They are given by:*

$$\begin{array}{llll} M_0 & : & e_0 & \mapsto 1, & T_{n_s} & \mapsto 0, & T_{n_{s'}} & \mapsto -1; \\ M_{q-1} & : & e_{q-1} & \mapsto 1, & T_{n_s} & \mapsto -1, & T_{n_{s'}} & \mapsto 0; \\ M_r & : & e_r & \mapsto 1, & T_{n_s} & \mapsto 0, & T_{n_{s'}} & \mapsto 0, \end{array}$$

where $0 < r < q - 1$.

In order to make use of the machinery of Hecke modules, we will need a precise relationship between smooth representations of $SU(1,1)(E/F)$ and $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ -modules. The following theorems provide us with the necessary link.

Theorem 3.6 ([1], Corollaire 6.1.10 (i)). *The functor of $I_S(1)$ -invariants $\pi \mapsto \pi^{I_S(1)}$ induces a bijection between isomorphism classes of smooth, irreducible, nonsupercuspidal representations of the group $SU(1,1)(E/F)$ and isomorphism classes of simple, finite-dimensional, nonsupersingular right $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ -modules.*

Theorem 3.7 ([1], Corollaire 6.1.10 (ii)). *The functor of $I_S(1)$ -invariants $\pi \mapsto \pi^{I_S(1)}$ induces a bijection between isomorphism classes of smooth, irreducible representations of the group $SU(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ and isomorphism classes of simple, finite-dimensional right $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ -modules. Under this bijection, supercuspidal representations correspond to supersingular modules.*

4. NONSUPERCUSPIDAL REPRESENTATIONS

Let \mathbb{G} be the F -rational points of a connected quasi-split reductive group of relative rank 1. The smooth, irreducible, nonsupercuspidal representations of \mathbb{G} have been classified by Abdellatif in [1], Chapitre 5. In particular, these results apply to the group $G = U(1,1)(E/F)$. We recall the results here.

Theorem 4.1 ([1], Théorème 5.1.2). *Let $\chi : T \longrightarrow \overline{\mathbb{F}}_p^\times$ be a smooth character of T , which we inflate to a character of B .*

- (1) *As a B -module, the $\overline{\mathbb{F}}_p$ -representation $\text{ind}_B^G(\chi)|_B$ is of length 2, with irreducible subquotients given by the character χ and the representation V_χ , consisting of elements of $\text{ind}_B^G(\chi)$ which take the value 0 at the identity.*
- (2) *The following are equivalent:*
 - (a) *The B -module $\text{ind}_B^G(\chi)|_B$ is semisimple;*
 - (b) *The G -module $\text{ind}_B^G(\chi)$ is reducible;*
 - (c) *The B -character χ extends to a character of G .*

(3) If χ extends to a character of G , then $\text{ind}_B^G(\chi)$ admits as subquotients the representation χ (as a subrepresentation) and the representation $\chi \otimes \text{St}_G$ (as a quotient). Here

$$\text{St}_G := \text{ind}_B^G(1_B)/1_G$$

denotes the Steinberg representation of G , and 1_B and 1_G denote the trivial characters of B and G , respectively. The exact sequence

$$0 \longrightarrow \chi \longrightarrow \text{ind}_B^G(\chi) \longrightarrow \chi \otimes \text{St}_G \longrightarrow 0$$

does not split.

This theorem shows that the irreducible nonsupercuspidal representations divide into three families. The next result demonstrates the lack of isomorphisms between these representations.

Theorem 4.2 ([1], Théorème 5.1.4, Lemme 5.1.5, Théorème 5.1.7 (2)). *There do not exist any isomorphisms between representations from distinct families. If χ and χ' are two characters of B which extend to G (resp. do not extend to G) and there exists an isomorphism $\chi \otimes \text{St}_G \cong \chi' \otimes \text{St}_G$ (resp. $\text{ind}_B^G(\chi) \cong \text{ind}_B^G(\chi')$), then $\chi = \chi'$.*

We make Theorem 4.1 explicit. For any finite extension L of F , we let ω denote the character of L^\times whose value at ϖ_L is 1, and whose restriction to \mathfrak{o}_L^\times is given by the composition

$$(2) \quad \omega : \mathfrak{o}_L^\times \xrightarrow{\mathfrak{r}_L} k_L^\times \xrightarrow{\iota} \overline{\mathbb{F}}_p^\times,$$

where $\mathfrak{r}_L : \mathfrak{o}_L \longrightarrow k_L$ denotes the reduction modulo the maximal ideal. For $\lambda \in \overline{\mathbb{F}}_p^\times$, we denote by $\mu_\lambda : L^\times \longrightarrow \overline{\mathbb{F}}_p^\times$ the unramified character taking the value λ at ϖ_L . In this notation, we have that any smooth character of E^\times (resp. F^\times) is of the form $\mu_\lambda \omega^r$ for a unique $\lambda \in \overline{\mathbb{F}}_p^\times$ and a unique $0 \leq r < q^2 - 1$ (resp. $0 \leq r < q - 1$). Likewise, any smooth character of $\text{U}(1)(E/F)$ is of the form ω^r for a unique $0 \leq r < q + 1$.

Since $T \cong E^\times$, we will identify the smooth characters of T (and B) with those of E^\times , by the formula

$$\mu_\lambda \omega^r \left(\begin{pmatrix} a & b \\ 0 & \bar{a}^{-1} \end{pmatrix} \right) := \mu_\lambda \omega^r(a).$$

Assume now that the character $\mu_\lambda \omega^r$ extends to a character of G . This extension must therefore be trivial on the derived subgroup G_S and its maximal torus T_S . In particular, this implies

$$\begin{aligned} \lambda &= \mu_\lambda \omega^r(\varpi) = 1, \\ a^r &= \mu_\lambda \omega^r([a]) = 1, \end{aligned}$$

where $a \in \mathbb{F}_q^\times$. Hence, we see that if the character $\mu_\lambda \omega^r$ extends to G , we must have $\lambda = 1$ and $r = (q - 1)m$ for $0 \leq m < q + 1$. The converse statement is easily verified, and combining Theorems 4.1 and 4.2, we obtain the following theorem.

Theorem 4.3. *Let π be a smooth, irreducible, nonsupercuspidal representation of the group $\text{U}(1, 1)(E/F)$. Then π is isomorphic to one and only one of the following representations:*

- the smooth $\overline{\mathbb{F}}_p$ -characters $\omega^k \circ \det$, where $0 \leq k < q + 1$;
- twists of the Steinberg representation $\omega^k \circ \det \otimes \text{St}_G$, where $0 \leq k < q + 1$;
- the principal series representations $\text{ind}_B^G(\mu_\lambda \omega^r)$, where $\lambda \in \overline{\mathbb{F}}_p^\times$ and $0 \leq r < q^2 - 1$ with $(r, \lambda) \neq ((q - 1)m, 1)$.

In addition to this classification, we will also need to know how the nonsupercuspidal representations behave upon restriction to $SU(1,1)(E/F)$.

Theorem 4.4. *We have the following isomorphisms:*

- (a) $\omega^k \circ \det|_{G_S} \cong 1_{G_S}$, where 1_{G_S} denotes the trivial character of G_S ;
- (b) $\text{ind}_B^G(\mu_\lambda \omega^r)|_{G_S} \cong \text{ind}_{B_S}^{G_S}(\mu_\lambda \omega^{r'})$, where r' denotes the unique integer such that $0 \leq r' < q-1$ and $r' \equiv r \pmod{q-1}$;
- (c) $\omega^k \circ \det \otimes \text{St}_G|_{G_S} \cong \text{St}_{G_S}$, where $\text{St}_{G_S} := \text{ind}_{B_S}^{G_S}(1_{B_S})/1_{G_S}$ is the Steinberg representation of G_S .

Proof. We proceed as in the proof of Théorème 2.16 of [2]. Part (a) follows from the definition of G_S . For part (b), we note that $G = BG_S = G_S B$, and use the Mackey decomposition:

$$\begin{aligned} \text{ind}_B^G(\mu_\lambda \omega^r)|_{G_S} &\cong \bigoplus_{g \in G_S \backslash G/B} \text{ind}_{gBg^{-1} \cap G_S}^{G_S}((\mu_\lambda \omega^r)^{g^{-1}}|_{gBg^{-1} \cap G_S}) \\ &= \text{ind}_{B_S}^{G_S}(\mu_\lambda \omega^r|_{B_S}) \\ &= \text{ind}_{B_S}^{G_S}(\mu_\lambda \omega^{r'}). \end{aligned}$$

This isomorphism gives $\omega^k \circ \det \otimes \text{ind}_B^G(1_B)|_{G_S} \cong \text{ind}_B^G(\omega^{(1-q)k})|_{G_S} \cong \text{ind}_{B_S}^{G_S}(1_{B_S})$. Taking the quotient of $\text{ind}_{B_S}^{G_S}(1_{B_S})$ by the subrepresentation consisting of constant functions induces an injection

$$\omega^k \circ \det \otimes \text{St}_G|_{G_S} = \text{ind}_B^G(\omega^{(1-q)k})/(\omega^k \circ \det)|_{G_S} \hookrightarrow \text{ind}_{B_S}^{G_S}(1_{B_S})/1_{G_S} = \text{St}_{G_S}.$$

Proposition 2.7 (*loc. cit.*) shows that St_{G_S} is irreducible as a representation of G_S , and therefore the injection is an isomorphism. This proves (c). \square

Corollary 4.5. *If π is a smooth irreducible nonsupercuspidal representation of $U(1,1)(E/F)$, then π remains irreducible and nonsupercuspidal upon restriction to $SU(1,1)(E/F)$.*

Proof. This follows from the two preceding Theorems and Proposition 2.7 of [2]. \square

In determining the representations of $U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, we shall need to know how supercuspidal representations behave upon restriction to $SU(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. We record two results in this direction.

Proposition 4.6. *Let π be a smooth irreducible representation of G . Then π admits a central character, and the restriction $\pi|_{G_S}$ is semisimple of length at most 2.*

Proof. We proceed using a method outlined in Lemma 2.4 of [22]. We first note that, since K_1 is a pro- p group, the vector space π^{K_1} is nonzero, and has an action of the group K . Let $\sigma \subset \pi^{K_1}$ be an irreducible Γ -subrepresentation; in particular, the center of K acts on σ by a character. The injection

$$\sigma \hookrightarrow \pi|_K$$

gives, by reciprocity, a surjection of $\text{c-ind}_K^G(\sigma)$ onto π . Since K contains the center Z , we conclude that π admits a central character.

The subgroup ZG_S is an index 2 subgroup of G ; the choice of ϑ implies that θ represents the nontrivial coset. Consider first the restriction $\pi|_{ZG_S}$, and suppose that it is reducible, with a nonzero proper subrepresentation τ . Let V_τ denote the underlying vector space of τ .

It then follows that the space $V_\tau + \theta.V_\tau$ is nonzero and stable by G , so must be all of π . Likewise, the space $V_\tau \cap \theta.V_\tau$ is stable by G , and hence must be 0. Thus we see that

$$\pi|_{ZG_S} \cong \tau \oplus \tau^\theta,$$

where τ^θ denotes the representation with the same underlying space as τ , with the action given by first conjugating an element of ZG_S by θ . Moreover, the same argument as above implies that τ must be irreducible as a representation of ZG_S . Since π admits a central character, the representations τ and τ^θ will remain irreducible upon further restricting to G_S . Therefore, we see that if π is a smooth irreducible representation of G , the restriction $\pi|_{G_S}$ is semisimple of length at most 2. \square

Proposition 4.7. *Let π be a smooth irreducible representation of G , and let $\tau \subset \pi|_{G_S}$ be a nonzero irreducible G_S -subrepresentation. Then π is supercuspidal if and only if τ is supercuspidal.*

Proof. Suppose π is not supercuspidal, so that π is isomorphic to either a character of G , a twist of the Steinberg representation, or a principal series representation. Theorem 4.4 implies that the restriction $\pi|_{G_S}$ must be the trivial character of G_S , the Steinberg representation of G_S , or an irreducible principal series for G_S , respectively. Therefore τ cannot be supercuspidal.

Suppose now that $\pi|_{G_S}$ contains a nonsupercuspidal representation τ . If τ is the trivial character of G_S , then Proposition 4.6 implies π is finite-dimensional. This implies by smoothness that π must be a character, and hence not supercuspidal. We may therefore assume that τ is a quotient of a parabolically induced representation, that is to say, there exists $\chi : B_S \rightarrow \overline{\mathbb{F}}_p^\times$ such that τ is a quotient of $\text{ind}_{B_S}^{G_S}(\chi)$. Let $\tilde{\chi} : ZB_S \rightarrow \overline{\mathbb{F}}_p^\times$ denote a character whose restriction to B_S is χ , and whose restriction to Z is the central character of π . Mackey theory now implies that

$$\text{Hom}_{ZG_S}(\text{ind}_{ZB_S}^{ZG_S}(\tilde{\chi}), \pi|_{ZG_S}) \neq 0.$$

Using Frobenius Reciprocity and transitivity of induction, we obtain

$$\begin{aligned} \text{Hom}_{ZG_S}(\text{ind}_{ZB_S}^{ZG_S}(\tilde{\chi}), \pi|_{ZG_S}) &\cong \text{Hom}_G(\text{ind}_{ZG_S}^G(\text{ind}_{ZB_S}^{ZG_S}(\tilde{\chi})), \pi) \\ &\cong \text{Hom}_G(\text{ind}_{ZB_S}^G(\tilde{\chi}), \pi) \\ &\cong \text{Hom}_G(\text{ind}_B^G(\text{ind}_{ZB_S}^B(\tilde{\chi})), \pi) \\ &\cong \text{Hom}_G(\text{ind}_B^G(\chi' \oplus \chi''), \pi), \end{aligned}$$

where χ' and χ'' are the two extensions of $\tilde{\chi}$ to B . From this we see that π is a quotient of a parabolically induced representation, and therefore not supercuspidal. \square

5. SUPERCUSPIDAL REPRESENTATIONS

We suppose from this point onwards that $F = \mathbb{Q}_p$, $E = \mathbb{Q}_{p^2}$, and $\varpi = p$. We let \mathbb{Z}_p and \mathbb{Z}_{p^2} denote the rings of integers of \mathbb{Q}_p and \mathbb{Q}_{p^2} , respectively.

The supercuspidal representations of $\text{GL}_2(\mathbb{Q}_p)$ and $\text{SL}_2(\mathbb{Q}_p)$ have been classified by Breuil and Abdellatif, respectively (cf. [8], [2]). We review their results here.

5.1. The Group $GL_2(\mathbb{Q}_p)$. Let $0 \leq r \leq p-1$ be an integer. Denote by $\sigma_r = \text{Sym}^r(\overline{\mathbb{F}}_p^2)$ the $\overline{\mathbb{F}}_p$ -vector space of homogeneous polynomials in two variables of degree r , with an action of $\mathbb{Q}_p^\times GL_2(\mathbb{Z}_p)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^{r-i} y^i = (\mathfrak{r}_{\mathbb{Q}_p}(a)x + \mathfrak{r}_{\mathbb{Q}_p}(c)y)^{r-i} (\mathfrak{r}_{\mathbb{Q}_p}(b)x + \mathfrak{r}_{\mathbb{Q}_p}(d)y)^i,$$

$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \cdot x^{r-i} y^i = x^{r-i} y^i,$$

where $a, b, c, d \in \mathbb{Z}_p$, where \mathbb{Q}_p^\times denotes the center of $GL_2(\mathbb{Q}_p)$, and where $\mathfrak{r}_{\mathbb{Q}_p} : \mathbb{Z}_p \rightarrow \mathbb{F}_p$ is the reduction map. Proposition 8 of [3] shows that the algebra of $GL_2(\mathbb{Q}_p)$ -equivariant endomorphisms of the compactly induced representation $\text{c-ind}_{\mathbb{Q}_p^\times GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)}(\sigma_r)$ is isomorphic to a polynomial algebra over $\overline{\mathbb{F}}_p$ in one variable, generated by an endomorphism denoted T_r . For $\lambda \in \overline{\mathbb{F}}_p$ and χ a smooth character of \mathbb{Q}_p^\times , we denote by $\pi(r, \lambda, \chi)$ the representation of $GL_2(\mathbb{Q}_p)$ afforded by the cokernel of the map $T_r - \lambda$, twisted by χ :

$$\pi(r, \lambda, \chi) := \chi \circ \det \otimes \frac{\text{c-ind}_{\mathbb{Q}_p^\times GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)}(\sigma_r)}{(T_r - \lambda)}.$$

The necessary properties of the representations $\pi(r, \lambda, \chi)$ are summarized in the following theorem.

Theorem 5.1. *In the following, r denotes an integer $0 \leq r \leq p-1$ and $\chi : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{F}}_p^\times$ a smooth character.*

- (a) *Every smooth irreducible supercuspidal representation of $GL_2(\mathbb{Q}_p)$ which admits a central character is of the form $\pi(r, 0, \chi)$.*
- (b) *The only isomorphisms among the representations $\pi(r, 0, \chi)$ are the following:*

$$\begin{aligned} \pi(r, 0, \chi) &\cong \pi(r, 0, \chi\mu_{-1}) \\ \pi(r, 0, \chi) &\cong \pi(p-1-r, 0, \chi\omega^r) \\ \pi(r, 0, \chi) &\cong \pi(p-1-r, 0, \chi\mu_{-1}\omega^r) \end{aligned}$$

- (c) *Let $Iw(1)$ denote the standard upper pro- p -Iwahori subgroup of $GL_2(\mathbb{Q}_p)$. We have*

$$\pi(r, 0, 1)^{Iw(1)} = \overline{\mathbb{F}}_p[\text{id}, x^r] \oplus \overline{\mathbb{F}}_p[\beta, x^r],$$

where, for $g \in GL_2(\mathbb{Q}_p)$ and $v \in \sigma_r$, $[g, v]$ denotes the element of $\text{c-ind}_{\mathbb{Q}_p^\times GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)}(\sigma_r)$ with support $\mathbb{Q}_p^\times GL_2(\mathbb{Z}_p)g^{-1}$ and value v at g , and $\overline{[g, v]}$ denotes its image in $\pi(r, 0, 1)$.

Proof. This follows from Theorems 33, 34 and Corollary 36 of [3], and Théorème 3.2.4 and Corollaires 4.1.1, 4.1.4, and 4.1.5 of [8]. We remark that the hypothesis of having a central character in part (a) may be omitted (cf. [6]). \square

5.2. The Group $SL_2(\mathbb{Q}_p)$. Consider now the group $SL_2(\mathbb{Q}_p)$. Théorème 3.36 of [2] implies that for any smooth irreducible representation σ of $SL_2(\mathbb{Q}_p)$, there exists a smooth irreducible representation Σ of $GL_2(\mathbb{Q}_p)$ such that σ is a Jordan-Hölder factor of Σ . Moreover, Corollaires 3.38 and 3.41 (*loc. cit.*) imply that in order to classify supercuspidal representations of $SL_2(\mathbb{Q}_p)$, it suffices to compute the restriction of the representations $\pi(r, 0, 1)$. Let $\pi_{r,\infty}$

denote the $\mathrm{SL}_2(\mathbb{Q}_p)$ -subrepresentation of $\pi(r, 0, 1)|_{\mathrm{SL}_2(\mathbb{Q}_p)}$ generated by $\overline{[\mathrm{id}, x^r]}$, and let $\pi_{r,0}$ denote the $\mathrm{SL}_2(\mathbb{Q}_p)$ -subrepresentation of $\pi(r, 0, 1)|_{\mathrm{SL}_2(\mathbb{Q}_p)}$ generated by $\overline{[\beta, x^r]}$.

Theorem 5.2. *In the following, r denotes an integer $0 \leq r \leq p-1$.*

- (a) *We have $\pi(r, 0, 1)|_{\mathrm{SL}_2(\mathbb{Q}_p)} \cong \pi_{r,\infty} \oplus \pi_{r,0}$.*
- (b) *The representations $\pi_{r,0}$ and $\pi_{r,\infty}$ are smooth, irreducible, admissible, and supercuspidal.*
- (c) *Conversely, any smooth irreducible supercuspidal representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ is isomorphic to one of the form $\pi_{r,0}$ or $\pi_{r,\infty}$.*
- (d) *The only isomorphism among the representations $\pi_{r,0}$ and $\pi_{r,\infty}$ is the following:*

$$\pi_{r,\infty} \cong \pi_{p-1-r,0}$$

- (e) *Let $Iw_S(1)$ denote the standard upper pro- p -Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. We have*

$$\begin{aligned} \pi_{r,\infty}^{Iw_S(1)} &= \overline{\mathbb{F}_p[\mathrm{id}, x^r]}, \\ \pi_{r,0}^{Iw_S(1)} &= \overline{\mathbb{F}_p[\beta, x^r]}. \end{aligned}$$

Proof. This follows from Propositions 4.5 and 4.7 and Corollaires 4.8 and 4.9 of [2], and the comments following Corollaire 4.9. \square

Definition 5.3. We let π_r denote the representation $\pi_{r,\infty}$. In light of Theorem 5.2, the representations π_r with $0 \leq r \leq p-1$ are a full set of representatives for the set of supercuspidal representations of $\mathrm{SL}_2(\mathbb{Q}_p)$.

We shall henceforth view the representations π_r as representations of $\mathrm{SU}(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ via the isomorphism $\mathrm{SU}(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) \cong \mathrm{SL}_2(\mathbb{Q}_p)$. We may now be more precise about the structures of $\pi_r^{I_S(1)}$ as Hecke modules.

Lemma 5.4. *The bijection of Theorem 3.7 is given explicitly by*

$$\pi_r \longmapsto \pi_r^{I_S(1)} \cong M_r,$$

where M_r is the Hecke module defined in Proposition 3.5.

Proof. This follows from Propositions 6.3.50, 6.3.52, and 6.3.54 of [1]. \square

5.3. The Group $\mathrm{U}(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. We now proceed to examine the supercuspidal representations of $\mathrm{U}(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. For $n \geq 1$, let $\mathrm{U}(1)_n$ denote the group $\mathrm{U}(1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) \cap (1 + p^n \mathbb{Z}_{p^2})$. We begin with a simple lemma.

Lemma 5.5. *Let $n \geq 1$. Any element $x \in 1 + p^n \mathbb{Z}_{p^2}$ can be written uniquely as $x = yz$, with $y \in 1 + p^n \mathbb{Z}_p$ and $z \in \mathrm{U}(1)_n$. In other words, we have*

$$1 + p^n \mathbb{Z}_{p^2} \cong (1 + p^n \mathbb{Z}_p) \times \mathrm{U}(1)_n.$$

Proof. Consider the norm map:

$$\begin{aligned} N_{\mathbb{Q}_{p^2}/\mathbb{Q}_p} : \quad \mathbb{Q}_{p^2}^\times &\longrightarrow \mathbb{Q}_p^\times \\ x &\longmapsto x\overline{x} \end{aligned}$$

Since the extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ is unramified, the norm map restricts to a surjection

$$N_{\mathbb{Q}_{p^2}/\mathbb{Q}_p} : 1 + p^n \mathbb{Z}_{p^2} \longrightarrow 1 + p^n \mathbb{Z}_p$$

([27], Chapitre V, Proposition 3(a)). The kernel is precisely $U(1)_n$, so we obtain an exact sequence

$$1 \longrightarrow U(1)_n \longrightarrow 1 + p^n \mathbb{Z}_{p^2} \xrightarrow{N_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}} 1 + p^n \mathbb{Z}_p \longrightarrow 1.$$

By Proposition 6(b) of Chapitre IV and Lemme 2 of Chapitre V (*loc. cit.*), the map $x \mapsto x^2$ is an automorphism of $1 + p^n \mathbb{Z}_p$. Therefore, the map $x \mapsto x^{1/2}$ gives a section to the norm map in the short exact sequence above. The result now follows. \square

Remark. The Lemma above and its proof are both valid for any unramified quadratic extension of nonarchimedean local fields of odd residual characteristic.

Let G_0 denote the subgroup of G generated by G_S and the central subgroup

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in U(1)_1 \right\}.$$

Since $U(1)_1$ is a pro- p group, the set of smooth, irreducible representations of G_0 and G_S are in canonical bijection. A more useful description of G_0 is given by the following Lemma.

Lemma 5.6. *Denote by η the (surjective) composite homomorphism*

$$G \xrightarrow{\det} U(1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) \xrightarrow{\text{v}_{p^2}} U(1)(\mathbb{F}_{p^2}/\mathbb{F}_p).$$

Then $\ker(\eta) = G_0$.

Proof. Let $h_s : \mathbb{Q}_{p^2}^\times \longrightarrow T$ be the homomorphism given by

$$h_s(a) = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}.$$

Every element g of G can be written (not necessarily uniquely) as

$$g = h_s([a])h_s(x)g',$$

with $a \in \mathbb{F}_{p^2}^\times$, $x \in 1 + p\mathbb{Z}_{p^2}$ and $g' \in G_S$. Assume that $g \in \ker(\eta)$, so that

$$a^{1-p} \equiv 1 \pmod{1 + p\mathbb{Z}_{p^2}}.$$

This implies $a \in \mathbb{F}_p^\times$ and $h_s([a]) \in G_S$. Hence, if g is in $\ker(\eta)$, then it must be of the form $h_s(x)g'$ for $x \in 1 + p\mathbb{Z}_{p^2}$ and $g' \in G_S$. Lemma 5.5 implies that $g \in G_0$, so $\ker(\eta)$ is a subgroup of G_0 . The reverse inclusion is easily verified. \square

By the Lemma above, we have $G/G_0 \cong U(1)(\mathbb{F}_{p^2}/\mathbb{F}_p)$. We may and do choose coset representatives $\{\delta_i\}_{i=0}^p$ such that $\delta_i \in T_0$; in particular, each δ_i normalizes the subgroups $I(1) \cap U$, $I(1) \cap U^-$ and $I(1) \cap T_S$.

Let $0 \leq r \leq p-1$, and let π_r be a smooth irreducible supercuspidal representation of G_S , inflated to G_0 . For δ_i a coset representative of G/G_0 as above, we let $\pi_r^{\delta_i}$ denote the representation with the same underlying space as π_r , with the action given by first conjugating an element of G_0 by δ_i . Since δ_i normalizes the subgroups $I(1) \cap U$, $I(1) \cap U^-$ and $I(1) \cap T_S$, we have $\pi_r^{I_S(1)} = (\pi_r^{\delta_i})^{I_S(1)}$ as vector spaces, and equation (1) shows that the actions of the operators T_{n_s} , $T_{n_{s'}}$ and $e_{r'}$ on these two spaces are the same. By Theorem 3.3 we have

$\pi_r^{I_S(1)} \cong (\pi_r^{\delta_i})^{I_S(1)}$ as $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ -modules, and Theorem 3.7 now implies that $\pi_r \cong \pi_r^{\delta_i}$ as G_S -representations (and consequently $\pi_r \cong \pi_r^{\delta_i}$ as G_0 -representations). Therefore, we may lift π_r to a projective representation of G . Since

$$H^2(G/G_0, \overline{\mathbb{F}}_p^\times) = \widehat{H}^0(G/G_0, \overline{\mathbb{F}}_p^\times) = 0,$$

this representation lifts to a genuine representation $\widetilde{\pi}_r$ of G .

It remains to determine the action of G on the lift $\widetilde{\pi}_r$ of π_r . Let h_s denote the homomorphism defined in the proof of Lemma 5.6. Since the image under h_s of $U(1)_1$ acts trivially on $\widetilde{\pi}_r$, we have $\widetilde{\pi}_r^{I(1)} = \widetilde{\pi}_r^{I_S(1)} = \overline{\mathbb{F}}_p v_r$, where $v_r = [\text{id}, x^r]$. The elements $h_s([a])$ for $a \in \mathbb{F}_{p^2}^\times$ normalize $I(1)$, so we have

$$h_s([a]).v_r = a^m v_r,$$

for $0 \leq m < p^2 - 1$. Since $a^{p+1} \in \mathbb{F}_p^\times$ for $a \in \mathbb{F}_{p^2}^\times$, we have

$$a^{(p+1)r} v_r = h_s([a]^{p+1}).v_r = a^{(p+1)m} v_r,$$

by definition of the representation π_r . Thus, we must have $m = r + (1-p)k$ for some $k \in \mathbb{Z}$. This leads to the following definition.

Definition 5.7. Let $0 \leq r \leq p-1$ and $0 \leq k < p+1$. We define the representation $\omega^k \circ \det \otimes \pi_r$ of $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ by the following conditions:

- $\omega^k \circ \det \otimes \pi_r|_{G_S} = \pi_r$;
- $(\omega^k \circ \det \otimes \pi_r)^{I(1)} = \pi_r^{I_S(1)} = \overline{\mathbb{F}}_p v_r$ as vector spaces;
- $h_s([a]).v_r = a^{r+(1-p)k} v_r$ for $a \in \mathbb{F}_{p^2}^\times$.

The preceding discussion ensures that the vector spaces $\omega^k \circ \det \otimes \pi_r$ are bona fide representations of G . We collect their properties in the following Proposition.

Proposition 5.8. Let $0 \leq r \leq p-1$ and $0 \leq k < p+1$.

- (a) The representations $\omega^k \circ \det \otimes \pi_r$ are smooth, irreducible, admissible, and supercuspidal.
- (b) The representations $\omega^k \circ \det \otimes \pi_r$ are pairwise nonisomorphic.

Proof. (a) The claim about irreducibility follows from the fact that the restriction of the representation $\omega^k \circ \det \otimes \pi_r$ to G_S is irreducible. The space of $I(1)$ -invariants is one-dimensional, and therefore $\omega^k \circ \det \otimes \pi_r$ is admissible. Proposition 4.7 implies that the representations are supercuspidal.

(b) Suppose that

$$\varphi : \omega^k \circ \det \otimes \pi_r \longrightarrow \omega^{k'} \circ \det \otimes \pi_{r'}$$

is a G -equivariant isomorphism. The map φ then defines a G_S -equivariant isomorphism, so we must have $r = r'$ by Theorem 5.2. Moreover, φ defines an isomorphism

$$\varphi : (\omega^k \circ \det \otimes \pi_r)^{I(1)} \longrightarrow (\omega^{k'} \circ \det \otimes \pi_{r'})^{I(1)};$$

the action of T_0 on these spaces implies that $k = k'$. □

Theorem 5.9. Let π be a smooth irreducible supercuspidal representation of the group $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. Then π is isomorphic to a representation of the form $\omega^k \circ \det \otimes \pi_r$ with $0 \leq r \leq p-1$ and $0 \leq k < p+1$.

Proof. Let π be a smooth irreducible supercuspidal representation. Propositions 4.6 and 4.7 imply that upon restriction to G_S , we must have

$$\pi|_{G_S} \cong \pi_r \quad \text{or} \quad \pi|_{G_S} \cong \pi_r \oplus \pi_r^\theta,$$

for some $0 \leq r \leq p-1$. Assume the latter. Since θ normalizes the subgroups $I_S(1) \cap U$, $I_S(1) \cap U^-$ and $I_S(1) \cap T$, equation (1) and Theorem 3.3 imply that the Hecke modules $\pi_r^{I_S(1)}$ and $(\pi_r^\theta)^{I_S(1)}$ are isomorphic. Theorem 3.7 now implies that $\pi_r^\theta \cong \pi_r$ as G_S -representations. Let v be a nonzero eigenvector for T_0 contained in $\pi^{I_S(1)} \cong \pi_r^{I_S(1)} \oplus (\pi_r^\theta)^{I_S(1)}$. The space $\langle G.v \rangle_{\overline{\mathbb{F}}_p}$ is stable by G , and therefore must be all of π . This implies that $\pi|_{G_S} \cong \pi_r$, a contradiction.

We may therefore assume that $\pi|_{G_S} \cong \pi_r$. The discussion preceding Definition 5.7 shows that there exists an integer k such that $\pi \cong \omega^k \circ \det \otimes \pi_r$. \square

Corollary 5.10. *Let π be a smooth irreducible representation of $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. Then π admits a central character and is admissible. Moreover, π is isomorphic to one and only one of the following representations:*

- the smooth $\overline{\mathbb{F}}_p$ -characters $\omega^k \circ \det$, where $0 \leq k < p+1$;
- twists of the Steinberg representation $\omega^k \circ \det \otimes \text{St}_G$, where $0 \leq k < p+1$;
- the principal series representations $\text{ind}_B^G(\mu_\lambda \omega^r)$, where $\lambda \in \overline{\mathbb{F}}_p^\times$ and $0 \leq r < p^2-1$ with $(r, \lambda) \neq ((q-1)m, 1)$;
- the supercuspidal representations $\omega^k \circ \det \otimes \pi_r$, where $0 \leq r \leq p-1$ and $0 \leq k < p+1$.

Proof. If π is not supercuspidal, then the result follows from Theorem 4.3, and if π is supercuspidal it follows from Proposition 5.8 and Theorem 5.9. It only remains to prove that no supercuspidal representation is isomorphic to a nonsupercuspidal representation. Assume this is the case; we then obtain a G_S -equivariant isomorphism between a supercuspidal representation and a nonsupercuspidal representation, contradicting Corollaire 3.19 of [2]. \square

5.4. L -packets. We define the *general unitary group* $\text{GU}(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ by

$$\{g \in \text{GL}_2(\mathbb{Q}_{p^2}) : g^* s g = \kappa s \text{ for some } \kappa \in \mathbb{Q}_p^\times\}.$$

The association $g \mapsto \kappa$ is in fact a character, and induces a surjective homomorphism $\text{sim} : \text{GU}(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) \rightarrow \mathbb{Q}_p^\times$. We obtain a short exact sequence of groups

$$1 \rightarrow U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) \rightarrow \text{GU}(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) \xrightarrow{\text{sim}} \mathbb{Q}_p^\times \rightarrow 1;$$

this exact sequence splits, and we have

$$\text{GU}(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p) \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{Q}_p^\times \right\}.$$

Since the group $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ is a normal subgroup of $\text{GU}(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, the latter group acts on G by conjugation, and consequently acts on representations of G . The following definition is adapted from the complex case (see Section 11.1 of [25]).

Definition 5.11. An L -packet of semisimple representations on $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ is a $\text{GU}(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ -orbit of smooth semisimple representations of G . An L -packet is called *supercuspidal* if it consists entirely of irreducible supercuspidal representations.

Proposition 5.12. *Let Π be an L -packet of smooth irreducible representations on $G = \mathrm{U}(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. Then Π has cardinality 1 if and only if it contains an irreducible non-supercuspidal representation. If Π is a supercuspidal L -packet, then it is of the form*

$$\Pi = \{\omega^k \circ \det \otimes \pi_r, \omega^{k+r+1} \circ \det \otimes \pi_{p-1-r}\},$$

for some $0 \leq r \leq p-1$, $0 \leq k < p+1$.

Proof. Assume first that Π contains an irreducible nonsupercuspidal representation π , and let

$$t = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

with $a \in \mathbb{Z}_p^\times$. Consider the representation π^t obtained by conjugation by t . Since

$$\pi^t|_{G_S} = (\pi|_{G_S})^t$$

and t normalizes $I_S(1)$, we have $((\pi|_{G_S})^t)^{I_S(1)} = (\pi|_{G_S})^{I_S(1)}$ as vector spaces. Equation (1) implies that the action of $\mathcal{H}_{\overline{\mathbb{F}}_p}(G_S, I_S(1))$ on these spaces is the same, and Theorem 3.6 shows that $\pi^t|_{G_S} \cong \pi|_{G_S}$. Theorem 4.4 now implies $\pi^t \cong \omega^k \circ \det \otimes \pi$ for some $0 \leq k < p+1$, and upon examining the action of the torus T , we conclude $\pi^t \cong \pi$.

Consider now the element

$$\beta s = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

Proposition 2.8 of [2] implies $\pi^{\beta s}|_{G_S} \cong \pi|_{G_S}$, and we proceed as above to conclude $\pi^{\beta s} \cong \pi$. These arguments imply that $\Pi = \{\pi\}$.

Assume now that Π contains a supercuspidal representation $\pi = \omega^k \circ \det \otimes \pi_r$, and let t be as above. The arguments of the first paragraph carry over (this time appealing to Theorem 3.7), and we conclude $\pi^t \cong \pi$.

To conclude the proof, we must compute $\pi^{\beta s}$; since $s \in \mathrm{U}(1, 1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, we have $\pi^s \cong \pi$ and it suffices to determine π^β . Corollaire 4.6 of [2] implies

$$\pi^\beta|_{G_S} \cong \pi_r^\beta \cong \pi_{p-1-r},$$

and Theorem 5.9 gives $\pi^\beta \cong \omega^{k'} \circ \det \otimes \pi_{p-1-r}$ for some k' . Now, the element β normalizes $I(1)$, and therefore we have

$$\overline{\mathbb{F}}_p v_r = \pi^{I(1)} = (\pi^\beta)^{I(1)}$$

as vector spaces. For $a \in \mathbb{F}_{p^2}^\times$, the action of $h_s([a])$ on $(\pi^\beta)^{I(1)}$ is given by

$$v_r \longmapsto a^{-pr-pk+k} v_r,$$

while the action of $h_s([a])$ on $(\omega^{k'} \circ \det \otimes \pi_{p-1-r})^{I(1)}$ is given by

$$v_{p-1-r} \longmapsto a^{p-1-r+(1-p)k'} v_{p-1-r}.$$

Since these characters coincide, we have that $k' \equiv r + k + 1 \pmod{p+1}$. This implies

$$\Pi = \{\omega^k \circ \det \otimes \pi_r, \omega^{k+r+1} \circ \det \otimes \pi_{p-1-r}\},$$

which concludes the proof. \square

6. GALOIS GROUPS AND REPRESENTATIONS

In this section we recall the definitions associated to Galois representations attached to unitary groups.

6.1. Galois Groups. Let $\mathcal{G}_{\mathbb{Q}_p} := \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ denote the absolute Galois group of \mathbb{Q}_p , and $\mathcal{I}_{\mathbb{Q}_p}$ the inertia subgroup of elements which act trivially on the residue field $k_{\overline{\mathbb{Q}_p}}$. For any extension F of \mathbb{Q}_p , contained in $\overline{\mathbb{Q}_p}$, we define $\mathcal{G}_F := \text{Gal}(\overline{\mathbb{Q}_p}/F)$. Let \mathbb{Q}_p^{ur} denote the maximal unramified extension of \mathbb{Q}_p ; we may then realize the subgroup $\mathcal{I}_{\mathbb{Q}_p}$ as

$$\mathcal{I}_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{ur}).$$

This gives $\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p} \cong \text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \cong \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$, where the last isomorphism is given by sending the arithmetic (or geometric) Frobenius to 1.

For $n \geq 1$, we let \mathbb{Q}_{p^n} denote the unique unramified extension of \mathbb{Q}_p of degree n contained in $\overline{\mathbb{Q}_p}$, and let

$$(3) \quad \iota_n : \mathbb{Q}_{p^n}^\times \longrightarrow \mathcal{G}_{\mathbb{Q}_{p^n}}^{ab}$$

denote the reciprocity map of local class field theory, normalized so that uniformizers correspond to geometric Frobenius elements. We shall denote by Fr_p a fixed element of $\mathcal{G}_{\mathbb{Q}_p}$ whose image in $\mathcal{G}_{\mathbb{Q}_p}^{ab}$ is equal to $\iota_1(p^{-1})$.

Using the injections ι_n , we will identify the smooth $\overline{\mathbb{F}_p}$ -characters of $\mathbb{Q}_{p^n}^\times$ and $\mathcal{G}_{\mathbb{Q}_{p^n}}$ in the following way. We fix a compatible system $\{p^{n-1}\sqrt[p]{p}\}_{n \geq 1}$ of $(p^n - 1)^{\text{th}}$ roots of p , and let $\omega_n : \mathcal{I}_{\mathbb{Q}_p} \longrightarrow \overline{\mathbb{F}_p}^\times$ denote the character given by

$$(4) \quad \omega_n : h \longmapsto \iota \circ \mathbf{r}_{\overline{\mathbb{Q}_p}} \left(\frac{h \cdot p^{n-1}\sqrt[p]{p}}{p^{n-1}\sqrt[p]{p}} \right),$$

where $h \in \mathcal{I}_{\mathbb{Q}_p}$ and $\mathbf{r}_{\overline{\mathbb{Q}_p}} : \overline{\mathbb{Z}_p} \longrightarrow k_{\overline{\mathbb{Q}_p}}$ denotes the reduction modulo the maximal ideal. Lemma 2.5 of [11] shows that the character ω_n extends to a character of $\mathcal{G}_{\mathbb{Q}_{p^n}}$; we continue to denote by ω_n the extension which sends the element Fr_p^n to 1.

Lemma 6.1. *For $n \geq 1$, we have $\omega_n \circ \iota_n = \omega$, where ω is defined in equation (2).*

Proof. Let

$$(\cdot, \cdot / \mathbb{Q}_{p^n}) : \mathbb{Q}_{p^n}^\times \longrightarrow \mathcal{G}_{\mathbb{Q}_{p^n}}^{ab}$$

denote the norm residue symbol of the field \mathbb{Q}_{p^n} , constructed in Chapitre XIII of [27]; with this notation we have $\iota_n(x) = (x^{-1}, \cdot / \mathbb{Q}_{p^n})$ for $x \in \mathbb{Q}_{p^n}^\times$. We also let $\nu_n : \mathbb{Q}_{p^n}^\times \longrightarrow \mathbb{Z}$ denote the normalized valuation of $\mathbb{Q}_{p^n}^\times$, and let

$$(\cdot, \cdot)_{\nu_n} : \mathbb{Q}_{p^n}^\times \times \mathbb{Q}_{p^n}^\times \longrightarrow \mu_{p^n-1}(\mathbb{Q}_{p^n}^\times)$$

denote the Hilbert symbol, where $\mu_{p^n-1}(\mathbb{Q}_{p^n}^\times)$ denotes the group of $(p^n - 1)^{\text{th}}$ roots of unity in $\mathbb{Q}_{p^n}^\times$ (cf. Chapitre XIV, *loc. cit.*).

Denote by \mathbb{Z}_{p^n} the ring of integers of \mathbb{Q}_{p^n} , and suppose $u \in \mathbb{Z}_{p^n}^\times$. Propositions 6 and 8 of Chapitre XIV (*loc. cit.*) imply

$$\frac{\iota_n(u) \cdot p^{n-1}\sqrt[p]{p}}{p^{n-1}\sqrt[p]{p}} = \frac{(u^{-1}, \cdot / \mathbb{Q}_{p^n}) \cdot p^{n-1}\sqrt[p]{p}}{p^{n-1}\sqrt[p]{p}} = (p, u^{-1})_{\nu_n} = \left[\mathbf{r}_{\overline{\mathbb{Q}_p}} \left((-1)^{\nu_n(p)} \frac{p^{\nu_n(u^{-1})}}{u^{-\nu_n(p)}} \right) \right] = [\mathbf{r}_{\overline{\mathbb{Q}_p}}(u)].$$

Applying $\iota \circ \mathbf{r}_{\overline{\mathbb{Q}}_p}$ to both sides shows that $\omega_n \circ \iota_n(u) = \omega(u)$.

The functorial properties of the reciprocity maps ι_n imply that we have equalities

$$\omega_n \circ \iota_n(p^{-1}) = \omega_n \circ \text{ver} \circ \iota_1(p^{-1}) = \omega_n(\text{Fr}_p^n) = 1 = \omega(p^{-1}),$$

where $\text{ver} : \mathcal{G}_{\overline{\mathbb{Q}}_p}^{ab} \longrightarrow \mathcal{G}_{\overline{\mathbb{Q}}_{p^n}}^{ab}$ denotes the transfer map. The result now follows. \square

Lemma 6.2. *For $h \in \mathcal{I}_{\overline{\mathbb{Q}}_p}$ and $n \geq 1$, we have*

$$\omega_n(\text{Fr}_p h \text{Fr}_p^{-1}) = \omega_n(h)^p.$$

Proof. See the proof of Lemma 2.5 in [11]. \square

For $\lambda \in \overline{\mathbb{F}}_p^\times$ and $n \geq 1$, we let $\mu_{n,\lambda} : \mathcal{G}_{\overline{\mathbb{Q}}_{p^n}} \longrightarrow \overline{\mathbb{F}}_p^\times$ denote the unramified character which is trivial on $\mathcal{I}_{\overline{\mathbb{Q}}_p}$ and sends Fr_p^n to λ .

Corollary 6.3. *Let $n \geq 1$. Every smooth $\overline{\mathbb{F}}_p$ -character of $\mathcal{G}_{\overline{\mathbb{Q}}_{p^n}}$ is of the form $\mu_{n,\lambda}\omega_n^r$, where $\lambda \in \overline{\mathbb{F}}_p^\times$ and $0 \leq r < p^n - 1$. Moreover, the reciprocity maps ι_n induce a bijection between smooth $\overline{\mathbb{F}}_p$ -characters of $\mathcal{G}_{\overline{\mathbb{Q}}_{p^n}}$ and $\mathbb{Q}_{p^n}^\times$, given explicitly by $\mu_{n,\lambda}\omega_n^r \circ \iota_n = \mu_{\lambda^{-1}}\omega^r$.*

6.2. L -groups. We now review the construction of L -groups for the group $\text{U}(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. For general references on L -groups, the reader should consult [7]; for the specific case of unitary groups, see Appendix A of [5], [23], or Section 1.8 of [25].

Let \widehat{G} denote the $\overline{\mathbb{F}}_p$ -valued points of the dual group of $G = \text{U}(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$; since G splits over \mathbb{Q}_{p^2} , we have $\widehat{G} = \text{GL}_2(\overline{\mathbb{F}}_p)$. Let α_1 be the $\overline{\mathbb{F}}_p$ -character of the diagonal maximal torus \widehat{T} of $\text{GL}_2(\overline{\mathbb{F}}_p)$ defined by

$$\alpha_1 \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = xy^{-1}$$

for $x, y \in \overline{\mathbb{F}}_p^\times$, and let \widehat{B} be the Borel subgroup of upper triangular matrices containing \widehat{T} . The root subgroup associated to α_1 (with respect to \widehat{B}) is given by

$$\mathfrak{X}_{\alpha_1} = \left\{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \overline{\mathbb{F}}_p \right\}.$$

Let $X_1 := u(1)$ be a basis for \mathfrak{X}_{α_1} . The discussion of [7], Section 1, implies we have $\text{Aut}(\widehat{G}) \cong \text{Inn}(\widehat{G}) \rtimes \Theta$, where Θ denotes the group of pinned automorphisms preserving the quadruple $\{\widehat{G}, \widehat{B}, \widehat{T}, X_1\}$. The group Θ is of order 2, generated by the automorphism

$$g \longmapsto \Phi_2(g^\top)^{-1}\Phi_2^{-1},$$

where

$$\Phi_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since the group G splits over \mathbb{Q}_{p^2} , the discussion of Section 1 (*loc. cit.*) implies we have an injection (indeed, an isomorphism) of $\text{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ into Θ .

Definition 6.4. The L -group of G is defined as the semidirect product

$${}^L G = \widehat{G} \rtimes \mathcal{G}_{\overline{\mathbb{Q}}_p} = \text{GL}_2(\overline{\mathbb{F}}_p) \rtimes \mathcal{G}_{\overline{\mathbb{Q}}_p},$$

with the action of $\mathcal{G}_{\mathbb{Q}_p}$ on \widehat{G} given by

$$\begin{aligned}\mathrm{Fr}_p g \mathrm{Fr}_p^{-1} &= \Phi_2(g^\top)^{-1} \Phi_2^{-1}, \\ h g h^{-1} &= g,\end{aligned}$$

for $g \in \widehat{G}$, $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

In addition to the group G , we shall also need unitary groups of lower rank. In particular, we will need the endoscopic group associated to G . For the general definition in the complex case, see Sections 4.2 and 4.6 of [25].

Definition 6.5. (a) The group $J := (U(1) \times U(1))(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ is the unique *elliptic endoscopic group* associated to $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$.

(b) The L -group of J is defined as the semidirect product

$${}^L J = \widehat{J} \rtimes \mathcal{G}_{\mathbb{Q}_p} = (\overline{\mathbb{F}}_p^\times \times \overline{\mathbb{F}}_p^\times) \rtimes \mathcal{G}_{\mathbb{Q}_p},$$

with the action of $\mathcal{G}_{\mathbb{Q}_p}$ on \widehat{J} given by

$$\begin{aligned}\mathrm{Fr}_p(x, y) \mathrm{Fr}_p^{-1} &= (x^{-1}, y^{-1}), \\ h(x, y) h^{-1} &= (x, y),\end{aligned}$$

for $x, y \in \overline{\mathbb{F}}_p^\times$, $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

(c) The L -group of $U(1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ is defined as the semidirect product

$${}^L U(1) = \overline{\mathbb{F}}_p^\times \rtimes \mathcal{G}_{\mathbb{Q}_p},$$

with the action of $\mathcal{G}_{\mathbb{Q}_p}$ on $\overline{\mathbb{F}}_p^\times$ given by

$$\begin{aligned}\mathrm{Fr}_p x \mathrm{Fr}_p^{-1} &= x^{-1}, \\ h x h^{-1} &= x,\end{aligned}$$

for $x \in \overline{\mathbb{F}}_p^\times$, $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

Proposition 6.6 ([25], Proposition 4.6.1). *There exists a homomorphism*

$$\xi : {}^L J \hookrightarrow {}^L G,$$

which commutes with the projections to $\mathcal{G}_{\mathbb{Q}_p}$, given by

$$\begin{aligned}(x, y) &\longmapsto \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \\ (1, 1) \mathrm{Fr}_p &\longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{Fr}_p, \\ (1, 1) h &\longmapsto \begin{pmatrix} \mu_{2,-1}(h) & 0 \\ 0 & \mu_{2,-1}(h) \end{pmatrix} h,\end{aligned}$$

where $x, y \in \overline{\mathbb{F}}_p^\times$, $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

6.3. Langlands Parameters for $U(1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. We begin by defining and investigating Langlands parameters in characteristic p associated to $U(1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$.

Definition 6.7. A *Langlands parameter* is a homomorphism

$$\varphi : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L U(1),$$

such that the composition of φ with the canonical projection ${}^L U(1) \longrightarrow \mathcal{G}_{\mathbb{Q}_p}$ is the identity map of $\mathcal{G}_{\mathbb{Q}_p}$. We say two Langlands parameters are *equivalent* if they are conjugate by an element of $\overline{\mathbb{F}}_p^\times$.

With this definition, we come to our first result.

Proposition 6.8. *Let $\varphi : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L U(1)$ be a Langlands parameter. Then there exists $0 \leq k < p + 1$ such that, up to equivalence, φ is of the following form:*

$$\begin{aligned} \text{Fr}_p &\longmapsto \text{Fr}_p \\ h &\longmapsto \omega_2^{(1-p)k}(h)h, \end{aligned}$$

where $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

Proof. The conjugation action of $\mathcal{G}_{\mathbb{Q}_p}$ on $\overline{\mathbb{F}}_p^\times$ shows that, up to equivalence, we have $\varphi(\text{Fr}_p) = \text{Fr}_p$. It remains to determine the image of $\mathcal{G}_{\mathbb{Q}_{p^2}}$. Since $\mathcal{G}_{\mathbb{Q}_{p^2}}$ acts trivially on $\overline{\mathbb{F}}_p^\times$, we see that φ must be of the form

$$\varphi(h) = \mu_{2,\lambda} \omega_2^r(h)h,$$

where $\lambda \in \overline{\mathbb{F}}_p^\times$, $0 \leq r < p^2 - 1$, and $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$. Examining the element $\varphi(\text{Fr}_p^2)$, we see that

$$\begin{aligned} \lambda \text{Fr}_p^2 &= \varphi(\text{Fr}_p^2) \\ &= \varphi(\text{Fr}_p)^2 \\ &= \text{Fr}_p^2, \end{aligned}$$

and thus $\lambda = 1$.

Lemma 6.2 now gives

$$\begin{aligned} \omega_2^{pr}(h) \text{Fr}_p h \text{Fr}_p^{-1} &= \varphi(\text{Fr}_p h \text{Fr}_p^{-1}) \\ &= \varphi(\text{Fr}_p) \varphi(h) \varphi(\text{Fr}_p)^{-1} \\ &= \text{Fr}_p \omega_2^r(h) h \text{Fr}_p^{-1} \\ &= \omega_2^{-r}(h) \text{Fr}_p h \text{Fr}_p^{-1}, \end{aligned}$$

which implies $r \equiv 0 \pmod{p-1}$. □

Definition 6.9. Let $0 \leq k < p + 1$. We denote by $\eta_k : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L U(1)$ the Langlands parameter constructed in Proposition 6.8; explicitly, we have

$$\begin{aligned} \eta_k(\text{Fr}_p) &= \text{Fr}_p, \\ \eta_k(h) &= \omega_2^{(1-p)k} h, \end{aligned}$$

where $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

Corollary 6.10. *There is a bijection between the Langlands parameters associated to the group $U(1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ and smooth irreducible representations of $U(1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, given explicitly by*

$$\eta_k \longleftrightarrow \omega^k,$$

where $0 \leq k < p + 1$, and ω is the character defined in equation (2).

6.4. Langlands Parameters for G . We now proceed to explore Langlands parameters in characteristic p for the group $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. For the analogous definitions in the complex setting, see [25], [26], and Appendix A of [5].

Definition 6.11. (a) A *Langlands parameter* is a homomorphism

$$\varphi : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L G,$$

such that the composition of φ with the canonical projection ${}^L G \longrightarrow \mathcal{G}_{\mathbb{Q}_p}$ is the identity map of $\mathcal{G}_{\mathbb{Q}_p}$. We say two Langlands parameters are *equivalent* if they are conjugate by an element of \widehat{G} .

(b) Let $\varphi : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L G$ be a Langlands parameter and let $0 \leq k < p+1$. We define the *twist* of φ by $\omega_2^{(1-p)k}$, denoted $\varphi \otimes \omega_2^{(1-p)k}$, by

$$\begin{aligned} \varphi \otimes \omega_2^{(1-p)k}(h) &= \varphi(h) \begin{pmatrix} \omega_2^{(1-p)k}(h) & 0 \\ 0 & \omega_2^{(1-p)k}(h) \end{pmatrix}, \\ \varphi \otimes \omega_2^{(1-p)k}(\text{Fr}_p h) &= \varphi(\text{Fr}_p h) \begin{pmatrix} \omega_2^{(1-p)k}(h) & 0 \\ 0 & \omega_2^{(1-p)k}(h) \end{pmatrix}, \end{aligned}$$

where $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$. One easily checks that this is well-defined and gives a bona fide Langlands parameter.

Definition 6.12. Let $\varphi : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L G$ be a Langlands parameter. Since the group $\mathcal{G}_{\mathbb{Q}_{p^2}}$ acts trivially on \widehat{G} , the restriction of φ to $\mathcal{G}_{\mathbb{Q}_{p^2}}$ must be of the form

$$\varphi(h) = \varphi_0(h)h,$$

where $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$ and $\varphi_0 : \mathcal{G}_{\mathbb{Q}_{p^2}} \longrightarrow \widehat{G}$ is a *homomorphism*. As $\widehat{G} = \text{GL}_2(\overline{\mathbb{F}}_p)$, φ_0 is a Galois representation; we call it the *Galois representation associated to φ* .

Definition 6.13. Let $\varphi : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L G$ be a Langlands parameter. We say φ is *stable* if the associated Galois representation $\varphi_0 : \mathcal{G}_{\mathbb{Q}_{p^2}} \longrightarrow \widehat{G}$ is irreducible.

Our first result on Langlands parameters for $U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ stands in stark contrast to the complex case (cf. [25], Section 15.1).

Proposition 6.14. *There do not exist any stable parameters $\varphi : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L G$.*

Proof. Suppose φ is a stable parameter, and let $\varphi_0 : \mathcal{G}_{\mathbb{Q}_{p^2}} \longrightarrow \widehat{G} = \text{GL}_2(\overline{\mathbb{F}}_p)$ denote the associated irreducible representation. It is well-known ([29], Section 1.14) that every two-dimensional irreducible mod- p representation of $\mathcal{G}_{\mathbb{Q}_{p^2}}$ is of the form

$$\text{ind}_{\mathcal{G}_{\mathbb{Q}_{p^4}}}^{\mathcal{G}_{\mathbb{Q}_{p^2}}}(\mu_{4,\lambda}\omega_4^m),$$

where $\lambda \in \overline{\mathbb{F}}_p^\times$, and $0 \leq m < p^4 - 1$ satisfies $m \not\equiv p^2 m \pmod{p^4 - 1}$. We assume φ_0 is of this form. In particular, we have

$$\varphi_0|_{\mathcal{G}_{\mathbb{Q}_{p^4}}} \cong \mu_{4,\lambda}\omega_4^m \oplus \mu_{4,\lambda}\omega_4^{p^2 m}$$

and

$$\det(\varphi_0) = \mu_{2,-1} \otimes (\mu_{4,\lambda}\omega_4^m) \circ \text{ver},$$

where $\text{ver} : \mathcal{G}_{\mathbb{Q}_{p^2}}^{ab} \longrightarrow \mathcal{G}_{\mathbb{Q}_{p^4}}^{ab}$ denotes the transfer map (the first fact follows from Mackey theory, while the second may be derived from Proposition 13.15 of [13]).

Let $g \in \widehat{G}$ be such that $\varphi(\text{Fr}_p) = g\text{Fr}_p$. We then have

$$\begin{aligned} \varphi_0(\text{Fr}_p^2)\text{Fr}_p^2 &= \varphi(\text{Fr}_p^2) \\ &= \varphi(\text{Fr}_p)^2 \\ &= g\text{Fr}_p g\text{Fr}_p \\ &= g\Phi_2(g^\top)^{-1}\Phi_2^{-1}\text{Fr}_p^2. \end{aligned}$$

Taking the determinant of the \widehat{G} -component gives

$$\begin{aligned} -\lambda &= \mu_{2,-1}(\text{Fr}_p^2)\mu_{4,\lambda}\omega_4^m(\text{Fr}_p^4) \\ &= \mu_{2,-1}(\text{Fr}_p^2)\mu_{4,\lambda}\omega_4^m \circ \text{ver}(\text{Fr}_p^2) \\ &= \det(g\Phi_2(g^\top)^{-1}\Phi_2^{-1}) \\ &= 1. \end{aligned}$$

This shows that, up to equivalence, we may assume φ_0 is of the following form:

$$\begin{aligned} \varphi_0(\text{Fr}_p^2) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \varphi_0(h) &= \begin{pmatrix} \mu_{4,-1}\omega_4^m(h) & 0 \\ 0 & \mu_{4,-1}\omega_4^{p^2m}(h) \end{pmatrix} \end{aligned}$$

for $h \in \mathcal{G}_{\mathbb{Q}_{p^4}}$.

Returning to the first set of equalities, we have $\varphi_0(\text{Fr}_p^2) = g\Phi_2(g^\top)^{-1}\Phi_2^{-1}$; a bit of algebra shows that g must be of the following form (up to equivalence by an element of the center of \widehat{G}):

$$g = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Let $h \in \mathcal{I}_{\mathbb{Q}_p}$. Using Lemma 6.2, we have

$$\varphi_0(\text{Fr}_p h \text{Fr}_p^{-1})\text{Fr}_p h \text{Fr}_p^{-1} = \begin{pmatrix} \omega_4^{pm}(h) & 0 \\ 0 & \omega_4^{p^3m}(h) \end{pmatrix} \text{Fr}_p h \text{Fr}_p^{-1}.$$

On the other hand, this is equal to

$$\begin{aligned} \varphi(\text{Fr}_p h \text{Fr}_p^{-1}) &= \varphi(\text{Fr}_p)\varphi(h)\varphi(\text{Fr}_p)^{-1} \\ &= g\text{Fr}_p\varphi_0(h)h\text{Fr}_p^{-1}g^{-1} \\ &= \begin{pmatrix} \frac{\omega_4^{-p^2m}(h)+\omega_4^{-m}(h)}{2} & \frac{-\omega_4^{-p^2m}(h)+\omega_4^{-m}(h)}{2} \\ \frac{-\omega_4^{-p^2m}(h)+\omega_4^{-m}(h)}{2} & \frac{\omega_4^{-p^2m}(h)+\omega_4^{-m}(h)}{2} \end{pmatrix} \text{Fr}_p h \text{Fr}_p^{-1}. \end{aligned}$$

This implies that $p^2m \equiv m \pmod{p^4-1}$, and we obtain a contradiction. \square

We have a completely analogous definition of Langlands parameters and equivalence for the endoscopic group J . We may classify such parameters in a manner nearly identical to that of Proposition 6.8.

Definition 6.15. Let $0 \leq k, \ell < p+1$. We denote by $\eta_{k,\ell} : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L J$ the Langlands parameter defined by

$$\begin{aligned}\eta_{k,\ell}(\mathrm{Fr}_p) &= (1, 1)\mathrm{Fr}_p, \\ \eta_{k,\ell}(h) &= (\omega_2^{(1-p)k}(h), \omega_2^{(1-p)\ell}(h))h,\end{aligned}$$

where $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

Corollary 6.16. *The Langlands parameters $\eta_{k,\ell}$ are pairwise inequivalent, and correspond bijectively (via the correspondence of Corollary 6.10) to the smooth irreducible representations of $J = (U(1) \times U(1))(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. Explicitly, the bijection is given by*

$$\eta_{k,\ell} \longleftrightarrow \omega^k \otimes \omega^\ell,$$

where $0 \leq k, \ell < p+1$.

Using the parameters $\eta_{k,\ell}$, we obtain the first nontrivial examples of Langlands parameters for the group G .

Definition 6.17. Let $0 \leq k, \ell < p+1$. We denote by $\varphi_{k,\ell} : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L G$ the Langlands parameter obtained by composing $\eta_{k,\ell}$ (of Definition 6.15) with ξ (of Proposition 6.6); explicitly, we have

$$\begin{aligned}\varphi_{k,\ell}(\mathrm{Fr}_p) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{Fr}_p, \\ \varphi_{k,\ell}(h) &= \begin{pmatrix} \mu_{2,-1}\omega_2^{(1-p)k}(h) & 0 \\ 0 & \mu_{2,-1}\omega_2^{(1-p)\ell}(h) \end{pmatrix} h,\end{aligned}$$

where $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

We say $\varphi_{k,\ell}$ is *regular* if $k \neq \ell$, and *singular* otherwise.

Lemma 6.18. *Let $0 \leq k, k', \ell, \ell' < p+1$. Then $\varphi_{k,\ell}$ is equivalent to $\varphi_{k',\ell'}$ if and only if the sets $\{k, \ell\}$ and $\{k', \ell'\}$ coincide.*

Proof. Conjugation by Φ_2 shows that $\varphi_{k,\ell}$ is equivalent to $\varphi_{\ell,k}$. Assume conversely that $\varphi_{k,\ell}$ is equivalent to $\varphi_{k',\ell'}$. Upon restricting to $\mathcal{G}_{\mathbb{Q}_{p^2}}$, we obtain

$$\mu_{2,-1}\omega_2^{(1-p)k} \oplus \mu_{2,-1}\omega_2^{(1-p)\ell} \cong \mu_{2,-1}\omega_2^{(1-p)k'} \oplus \mu_{2,-1}\omega_2^{(1-p)\ell'},$$

whence the result. \square

Corollary 6.19. *There exists a bijection between \widehat{G} -equivalence classes of regular Langlands parameters coming from the endoscopic group $J = (U(1) \times U(1))(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ and L -packets of irreducible supercuspidal representations on the group $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$, given by*

$$\varphi_{k,\ell} \longleftrightarrow \{\omega^\ell \circ \det \otimes \pi_{[k-\ell-1]}, \omega^k \circ \det \otimes \pi_{[\ell-k-1]}\},$$

where $0 \leq k, \ell < p+1$, and where $[k-\ell-1]$ (resp. $[\ell-k-1]$) denotes the unique integer between 0 and $p-1$ equivalent to $k-\ell-1$ (resp. $\ell-k-1$) modulo $p+1$. Moreover, this bijection is compatible with twisting on both sides (under the correspondence of Corollary 6.10).

Proof. Let $\Pi_{k,\ell}$ denote the L -packet on the right-hand side of the correspondence above. Proposition 5.12 and Lemma 6.18 show that two L -packets of the form $\Pi_{k,\ell}$ and $\Pi_{k',\ell'}$ are identical if and only if $\{k, \ell\} = \{k', \ell'\}$, if and only if $\varphi_{k,\ell}$ is equivalent to $\varphi_{k',\ell'}$. \square

Our next task will be to extend the correspondence of the above corollary. In doing so, we are led to consider Langlands parameters arising from a proper Levi subgroup of ${}^L G$. We let

$${}^L T = \widehat{T} \rtimes \mathcal{G}_{\mathbb{Q}_p},$$

where the action of $\mathcal{G}_{\mathbb{Q}_p}$ on \widehat{T} is the restriction of the action on \widehat{G} .

Proposition 6.20. *Let $\varphi : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L G$ be a Langlands parameter which factors through the group ${}^L T$, that is, such that φ is the composition*

$$\mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L T \hookrightarrow {}^L G,$$

where the second arrow denotes the canonical inclusion. Then there exist $0 \leq r < p^2 - 1$ and $\lambda \in \overline{\mathbb{F}}_p^\times$ such that, up to equivalence, φ is of the following form:

$$\begin{aligned} \mathrm{Fr}_p &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \mathrm{Fr}_p \\ h &\longmapsto \begin{pmatrix} \mu_{2,\lambda^{-1}} \omega_2^r(h) & 0 \\ 0 & \mu_{2,\lambda} \omega_2^{-pr}(h) \end{pmatrix} h, \end{aligned}$$

where $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

Proof. We proceed as in the proof of Proposition 6.8. Using the action of $\mathcal{G}_{\mathbb{Q}_p}$ on \widehat{T} we may assume that, up to equivalence, we have

$$\varphi(\mathrm{Fr}_p) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \mathrm{Fr}_p$$

for some $\lambda \in \overline{\mathbb{F}}_p^\times$. Let $\varphi_0 : \mathcal{G}_{\mathbb{Q}_{p^2}} \longrightarrow \widehat{T} \hookrightarrow \widehat{G}$ be the homomorphism associated to φ , so that

$$\varphi(h) = \varphi_0(h)h = \begin{pmatrix} \mu_{2,\lambda_1} \omega_2^{r_1}(h) & 0 \\ 0 & \mu_{2,\lambda_2} \omega_2^{r_2}(h) \end{pmatrix} h,$$

where $0 \leq r_i < p^2 - 1$, $\lambda_i \in \overline{\mathbb{F}}_p^\times$, and $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$. Consider the element $\varphi(\mathrm{Fr}_p^2)$; we have

$$\begin{aligned} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathrm{Fr}_p^2 &= \varphi(\mathrm{Fr}_p^2) \\ &= \varphi(\mathrm{Fr}_p)^2 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \mathrm{Fr}_p \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \mathrm{Fr}_p \\ &= \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \mathrm{Fr}_p^2, \end{aligned}$$

and thus $\lambda_2 = \lambda_1^{-1} = \lambda$.

Again using Lemma 6.2, we get

$$\begin{aligned} \begin{pmatrix} \omega_2^{pr_1}(h) & 0 \\ 0 & \omega_2^{pr_2}(h) \end{pmatrix} \mathrm{Fr}_p h \mathrm{Fr}_p^{-1} &= \varphi(\mathrm{Fr}_p h \mathrm{Fr}_p^{-1}) \\ &= \varphi(\mathrm{Fr}_p) \varphi(h) \varphi(\mathrm{Fr}_p)^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \mathrm{Fr}_p \begin{pmatrix} \omega_2^{r_1}(h) & 0 \\ 0 & \omega_2^{r_2}(h) \end{pmatrix} h \mathrm{Fr}_p^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \omega_2^{-r_2}(h) & 0 \\ 0 & \omega_2^{-r_1}(h) \end{pmatrix} \mathrm{Fr}_p h \mathrm{Fr}_p^{-1}, \end{aligned}$$

which shows that $r_2 \equiv -pr_1 \pmod{p^2 - 1}$. \square

Definition 6.21. Let $0 \leq r < p^2 - 1$ and $\lambda \in \overline{\mathbb{F}}_p^\times$. We denote by $\psi_{r,\lambda} : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow {}^L G$ the Langlands parameter constructed in Proposition 6.20; explicitly, we have

$$\begin{aligned} \psi_{r,\lambda}(\text{Fr}_p) &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \text{Fr}_p \\ \psi_{r,\lambda}(h) &= \begin{pmatrix} \mu_{2,\lambda^{-1}\omega_2^r}(h) & 0 \\ 0 & \mu_{2,\lambda}\omega_2^{-pr}(h) \end{pmatrix} h, \end{aligned}$$

where $h \in \mathcal{G}_{\mathbb{Q}_{p^2}}$.

We shall also need more precise information about equivalence classes of the Langlands parameters $\varphi_{k,\ell}$ and $\psi_{r,\lambda}$. This is the content of the following two Lemmas.

Lemma 6.22. Let $0 \leq r, r' < p^2 - 1$ and $\lambda, \lambda' \in \overline{\mathbb{F}}_p^\times$. Then $\psi_{r,\lambda}$ is equivalent to $\psi_{r',\lambda'}$ if and only if $r' = r$, $\lambda' = \lambda$ or $r' \equiv -pr \pmod{p^2 - 1}$, $\lambda' = \lambda^{-1}$.

Proof. Conjugating by the element

$$\begin{pmatrix} 0 & 1 \\ -\lambda^{-1} & 0 \end{pmatrix}$$

shows that $\psi_{r,\lambda}$ is equivalent to $\psi_{-pr,\lambda^{-1}}$. Assume conversely that $\psi_{r,\lambda}$ is equivalent to $\psi_{r',\lambda'}$. Upon restricting to $\mathcal{G}_{\mathbb{Q}_{p^2}}$, we obtain

$$\mu_{2,\lambda^{-1}\omega_2^r} \oplus \mu_{2,\lambda}\omega_2^{-pr} \cong \mu_{2,\lambda'^{-1}\omega_2^{r'}} \oplus \mu_{2,\lambda'}\omega_2^{-pr'},$$

which gives the desired result. \square

Lemma 6.23. Let $0 \leq k, \ell < p + 1$, $0 \leq r < p^2 - 1$, and $\lambda \in \overline{\mathbb{F}}_p^\times$. Then $\varphi_{k,\ell}$ is equivalent to $\psi_{r,\lambda}$ if and only if $k = \ell$, $r \equiv (1 - p)k \pmod{p^2 - 1}$, and $\lambda = -1$.

Proof. Fix an element $D \in \overline{\mathbb{F}}_p^\times$ such that $D^2 = -1$, and set

$$g = \begin{pmatrix} 1 & -D \\ 2^{-1} & 2^{-1}D \end{pmatrix}.$$

A bit of algebra verifies that $g\varphi_{k,k}(h)g^{-1} = \psi_{(1-p)k,-1}(h)$ for every $h \in \mathcal{G}_{\mathbb{Q}_p}$.

Assume conversely that $\varphi_{k,\ell}$ is equivalent to $\psi_{r,\lambda}$. Upon restricting to $\mathcal{G}_{\mathbb{Q}_{p^2}}$, we obtain

$$\mu_{2,-1}\omega_2^{(1-p)k} \oplus \mu_{2,-1}\omega_2^{(1-p)\ell} \cong \mu_{2,\lambda^{-1}\omega_2^r} \oplus \mu_{2,\lambda}\omega_2^{-pr};$$

this implies $\lambda = -1$, $r \equiv (1 - p)k \pmod{p^2 - 1}$ and $k = \ell$, which gives the result. \square

Definition 6.24. We define a “semisimple mod- p correspondence for $G = U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ ” to be the following correspondence between certain \widehat{G} -equivalence classes of Langlands parameters over $\overline{\mathbb{F}}_p$ and certain isomorphism classes of semisimple L -packets on $U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$:

- *the supercuspidal case:* Let $0 \leq k, \ell < p + 1$ with $k \neq \ell$.

$$\boxed{\varphi_{k,\ell} \longleftrightarrow \{\omega^\ell \circ \det \otimes \pi_{[k-\ell-1]}, \omega^k \circ \det \otimes \pi_{[\ell-k-1]}\}}$$

- *the nonsupercuspidal case:* Let $0 \leq r \leq p-1$, $\lambda \in \overline{\mathbb{F}}_p^\times$, and $0 \leq k < p+1$.
 - if $(r, \lambda) \neq (0, 1)$, $(p-1, 1)$:

$$\psi_{r,\lambda} \otimes \omega_2^{(1-p)k} = \psi_{r+(1-p)k,\lambda} \longleftrightarrow \left\{ \omega^k \circ \det \otimes \text{ind}_B^G(\mu_{\lambda^{-1}} \omega^{-pr}) \oplus \omega^k \circ \det \otimes \text{ind}_B^G(\mu_\lambda \omega^r) \right\}$$

- if $(r, \lambda) = (0, 1)$:

$$\psi_{0,1} \otimes \omega_2^{(1-p)k} = \psi_{(1-p)k,1} \longleftrightarrow \left\{ \omega^k \circ \det \oplus (\omega^k \circ \det \otimes \text{St}_G) \oplus \omega^k \circ \det \oplus (\omega^k \circ \det \otimes \text{St}_G) \right\}$$

6.5. Remarks.

- (1) Corollary 6.19 and Lemmas 6.18 and 6.22 imply that the correspondence above is well-defined.
- (2) We may state this correspondence more elegantly as follows. For $0 \leq r \leq p-1$, the group K acts irreducibly on the representation σ_r defined in Subsection 5.1. The spherical Hecke algebra $\mathcal{H}_{\overline{\mathbb{F}}_p}(G, K, \sigma_r)$ of G -equivariant endomorphisms of the compactly induced representation $\text{c-ind}_K^G(\sigma_r)$ is isomorphic to a polynomial algebra over $\overline{\mathbb{F}}_p$ in one variable, generated by an endomorphism τ_r . Explicitly, we have

$$\tau_r = \begin{cases} \tau_{r,1} & \text{if } r \neq 0, \\ \tau_{r,1} + 1 & \text{if } r = 0. \end{cases}$$

Here $\tau_{r,1}$ denotes the endomorphism of $\text{c-ind}_K^G(\sigma_r)$ which corresponds via Frobenius Reciprocity to the function with support $K\alpha^{-1}K$ and taking the value U_r at α^{-1} , where U_r is the endomorphism of σ_r given by

$$U_r \cdot x^{r-i} y^i = \begin{cases} 0 & \text{if } i \neq r, \\ y^r & \text{if } i = r. \end{cases}$$

For $\lambda \in \overline{\mathbb{F}}_p^\times$, we define

$$\pi(r, \lambda) := \frac{\text{c-ind}_K^G(\sigma_r)}{(\tau_r - \lambda)}.$$

A simple argument shows that

$$\pi(r, \lambda)|_{G_S} \cong \pi_0(r, \lambda),$$

where $\pi_0(r, \lambda)$ denotes the representation of $\text{SL}_2(\mathbb{Q}_p)$ (viewed as a representation of G_S) defined in [2], Section 3.4. Using Théorème 3.18 (*loc. cit.*) and the existence of certain $I(1)$ -invariant elements of $\pi(r, \lambda)$ (along with Theorem 4.4), we deduce

$$\pi(r, \lambda) \cong \begin{cases} \text{ind}_B^G(\mu_{\lambda^{-1}} \omega^{-pr}) & \text{if } (r, \lambda) \neq (0, 1), \\ \text{nonsplit extension of } 1_G \text{ by } \text{St}_G & \text{if } (r, \lambda) = (0, 1). \end{cases}$$

If we let π^{ss} denote the semisimplification of a smooth representation π of G , we obtain

$$\pi(r, \lambda)^{\text{ss}} = \begin{cases} \text{ind}_B^G(\mu_{\lambda^{-1}}\omega^{-pr}) & \text{if } (r, \lambda) \neq (0, 1), (p-1, 1), \\ \omega^p \circ \det \oplus (\omega^p \circ \det \otimes \text{St}_G) & \text{if } (r, \lambda) = (p-1, 1), \\ 1_G \oplus \text{St}_G & \text{if } (r, \lambda) = (0, 1). \end{cases}$$

The correspondence of Definition 6.24 now takes the form:

Definition 6.24, Modified.

- *The supercuspidal case:* Let $0 \leq k, \ell < p+1$ with $k \neq \ell$.

$$\varphi_{k,\ell} \longleftrightarrow \left\{ \omega^\ell \circ \det \otimes \pi_{[k-\ell-1]}, \omega^k \circ \det \otimes \pi_{[\ell-k-1]} \right\}$$

- *The nonsupercuspidal case:* Let $0 \leq r \leq p-1$, $\lambda \in \overline{\mathbb{F}}_p^\times$, and $0 \leq k < p+1$.

$$\psi_{r,\lambda} \otimes \omega_2^{(1-p)k} = \psi_{r+(1-p)k,\lambda} \longleftrightarrow \left\{ \omega^k \circ \det \otimes \pi(r, \lambda)^{\text{ss}} \oplus \omega^{k+r+1} \circ \det \otimes \pi(p-1-r, \lambda^{-1})^{\text{ss}} \right\}$$

- (3) The correspondences of Corollary 6.16 and Definition 6.24, along with the homomorphism ξ of Proposition 6.6, imply that we have an *endoscopic transfer map*

$$\tilde{\xi} : \mathfrak{Rep}_{\overline{\mathbb{F}}_p}((U(1) \times U(1))(\mathbb{Q}_{p^2}/\mathbb{Q}_p)) \longrightarrow \mathfrak{L}\text{-}\mathfrak{pack}_{\overline{\mathbb{F}}_p}(U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p))$$

from the category of smooth irreducible representations of $(U(1) \times U(1))(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ to the category of L -packets of semisimple representations on $U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. Using Lemma 6.23, the map $\tilde{\xi}$ is given explicitly by

$$\begin{aligned} \tilde{\xi}(\omega^k \otimes \omega^\ell) &= \left\{ \omega^\ell \circ \det \otimes \pi_{[k-\ell-1]}, \omega^k \circ \det \otimes \pi_{[\ell-k-1]} \right\} & \text{if } k \neq \ell, \\ \tilde{\xi}(\omega^k \otimes \omega^k) &= \left\{ \text{ind}_B^G(\mu_{-1}\omega^{(1-p)k}) \oplus \text{ind}_B^G(\mu_{-1}\omega^{(1-p)k}) \right\} & \text{if } k = \ell. \end{aligned}$$

This bears a striking resemblance to the complex case (see Proposition 11.1.1 of [25], especially points (c) and (e)). Moreover, the equation

$$\tilde{\xi}(1_J) = \tilde{\xi}(\omega^0 \otimes \omega^0) = \left\{ \text{ind}_B^G(\mu_{-1}) \oplus \text{ind}_B^G(\mu_{-1}) \right\}$$

gives an example of *transfer of unramified representations*, which may also be deduced from (a modified version of) the discussion in Section 2.7 of [23] (see also Theorem 4.4, *loc. cit.*, and Section 4.5 of [25]).

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